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# Three-body problem periodic orbits with vanishing angular momentum

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Periodic solutions of the general three-body problem are investigated in the shape space. Two different solutions are considered: the first is an extension of the well-known figure-eight orbit, and the second one is from the free-fall problem. Using the shape space, we reduce the dimension of the problem. These orbits are obtained numerically and described on the Euclidean plane and on the shape sphere.



## 1 Introduction

The general solution of the three-body problem is unknown. In fact, we have the Sundman solution that is very important for theoretical aspects, but it turns out to be useless in practice. Moreover, the angular momentum has to be nonzero in the Sundman solution. Special solutions of the three-body problem (without any restriction to masses) were discovered by Euler in 1765 and by Lagrange in 1772. In these solutions, all three bodies move along the Kepler trajectories while they are located on a straight line (the rectilinear Euler configuration) or at the vertices of an equilateral triangle (the Lagrange equilateral triangle). But when the angular momentum is equal to zero, the motion is homographic and terminates in a collision of the triple.

Early in the 20th century (Burrau 1913), the general three-body problem had been integrated and formulated as the so called Pythagorean problem. Here, three bodies are initially located at vertices of the Pythagorean triangle, have masses of 3, 4, and 5 units with opposite sides of the corresponding length, and all three bodies have zero initial velocities. In the 50's, using a regularization procedure (Schubart 1956), Schubart had found the periodic trajectories of the restricted three-body problem where two equal masses move along a line. In the 70's, some types of orbits for the general three-body problem were investigated numerically by Szebehely & Peters (1967), for the Pythagorean problem by Broucke & Lass (1973) and Broucke & Boggs (1975), and for periodic orbits by Henon (1976) and others.

The next strictly proved special solution was found more than two hundreds years after the appearing of the Lagrange equilateral orbits. This is the remarkable figure-eight trajectory by Chenciner & Montgomery (2000). Firstly, this orbit was found numerically by Moore (1993). Then in Chenciner & Montgomery (2000), it had been reopened, and the existence of the figure-eight form was proved using a variational method with symmetry constraints. In Barutello, Ferrario & Terracini (2004), it was shown that all finite symmetry groups of the Lagrangian action functional in the planar three-body problem include only ten items at all.

The dihedral group $D_6$ yields the figure-eight orbit that is called the simple choreography. Here, three equal masses move along the same closed curve lagged in phase from each other by one third of the period, $T/3$. Rather interesting and unexpected, the figure-eight orbit has zero angular momentum and is stable.

We consider the trajectories with zero angular momentum and these orbits as planar. Assuming that the barycenter is fixed at the origin of the coordinate system, we reduce the dimension of the configuration space down to 4. Lemaitre (1955) had reduced this dimension to 3 considering coordinates in space to be known as the shape one. Later in the 90's, a special geometric reduction was considered by Hsiang & Straume (1994) and Montgomery (1996). This is the "shape sphere", the sphere in the shape space, that Chenciner & Montgomery have used in their proof. In Moeckel & Montgomery (2013), the reduction, regularization, and blow-up of the planar three-body problem are described.

Following the paper by Chenciner & Montgomery (2000) a large number of works were devoted to using the variational method and searching special solutions of the $n$-body problem. A number of trajectories with vanishing angular momentum was found in Shuvakov & Dmitrasinovic (2013); for example, trajectories of symmetric periodic orbits for the three-body problem may be found in Titov (2006).

In this paper two different classes of orbits are considered: the split eight orbit and the free fall one. These orbits are essentially different; however, both orbits have zero angular momentum and, moreover, both types are described in the reduced space (the shape one) and on the Euclidean plane. This approach gives us the opportunity to make some

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conclusions about the properties of the obtained orbits and to reduce the space of the initial conditions to find the free-fall solutions.

## 2 Shape sphere

We can reduce the configuration space $Q$ of our problem considering all trajectories which coincide into one by being translated or rotated. To reduce the space $Q$ by translation, we can use the Jacobi coordinates

$$\begin{aligned}\boldsymbol{\xi}_1 &= \boldsymbol{r}_2 - \boldsymbol{r}_1,\\ \boldsymbol{\xi}_2 &= \boldsymbol{r}_3 - \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} = \boldsymbol{r}_3\frac{m_1+m_2+m_3}{m_1+m_2},\end{aligned}$$

and then use the mutual distances and the inertia moment

$$\begin{aligned}r_{12} &= |\boldsymbol{\xi}_1|,\\ r_{13} &= |\boldsymbol{\xi}_2 + \frac{m_2}{m_1+m_2}\boldsymbol{\xi}_1|,\\ r_{23} &= |\boldsymbol{\xi}_2 - \frac{m_1}{m_1+m_2}\boldsymbol{\xi}_1|,\\ I = r^2 &= \mu_1|\boldsymbol{\xi}_1|^2 + \mu_2|\boldsymbol{\xi}_2|^2.\end{aligned}$$

Here, $\mu_1 = \frac{m_1 m_2}{m_1+m_2}$, $\mu_2 = \frac{m_3(m_1+m_2)}{m_1+m_2+m_3}$. Now the configuration space has dimension 4. We get the same result by fixing the center of the masses at the origin $\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 = 0$.

*The next step.* Let us reduce our space by rotation using the Hopf mapping. Let $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$ be the complex points, then

$$\begin{aligned}w_1 &= \frac{1}{2}\mu_1|\boldsymbol{\xi}_1|^2 - \frac{1}{2}\mu_2|\boldsymbol{\xi}_2|^2,\\ w_2 + iw_3 &= \sqrt{\mu_1\mu_2}\,\boldsymbol{\xi}_1\bar{\boldsymbol{\xi}}_2,\\ w_1^2 + w_2^2 + w_3^2 &= \frac{1}{4}I^2 = \frac{1}{4}r^4.\end{aligned}$$

The space of $(w_1, w_2, w_3)$ is called the shape space and represents the space of the congruent triangles; here, $S = 2w/I$, $(s_1, s_2, s_3)$ is the shape sphere and it represents the space of the similar triangles.

The equator $s_3 = 0$ is the set of the degenerated triangles. On the equator, there are three Euler points and three binary collision points. All these ones lie on meridians of the isosceles triangles.

The relations between the $s_i$ and triangle sides are the following:

$$\begin{aligned}\tfrac{r_{12}^2}{r^2} &= \tfrac{m_1+m_2}{2m_1m_2}(1+s_1),\\ \tfrac{r_{13}^2}{r^2} &= \tfrac{m_1+m_3}{2m_1m_3} + \\ &\tfrac{m_2m_3 - m_1(m_1+m_2+m3)}{2m_1m_3(m_1+m_2)}s_1 + \tfrac{m_2\sqrt{m_1+m_2+m_3}}{(m_1+m_2)\sqrt{m_1m_2m_3}}s_2,\\ \tfrac{r_{23}^2}{r^2} &= \tfrac{m_2+m_3}{2m_2m_3} + \\ &\tfrac{m_1m_3 - m_2(m_1+m_2+m_3)}{2m_2m_3(m_1+m_2)}s_1 - \tfrac{m_1\sqrt{m_1+m_2+m_3}}{(m_1+m_2)\sqrt{m_1m_2m_3}}s_2.\end{aligned}$$

The third component $s_3$ can be obtained as follows:

$$s_3 = \pm\sqrt{1 - s_1^2 - s_2^2}\,.$$

Following Lemaitre (1955), we can introduce the polar coordinates $\varphi$, $\theta$ and write

$$r_{ij}^2 = r^2(1 - \cos\theta\cos(\varphi - \varphi_k)),$$

where $\varphi_k$ is the longitude of the $k$-th Euler point; $i$, $j$, and $k$ are permutation of the numbers 1, 2, and 3.

If the angular momentum vector is zero, we have two remarkable properties of trajectories (Chenciner & Montgomery 2000; Montgomery 2002):

- We can restore the real trajectory (up to rotation) if we have this trajectory on the shape sphere.
- Coordinates of the third component $s_3$ are monotonic functions between its two local extrema that lie on the opposite hemispheres.

Due to the scale symmetry, we can fix the period $T = 2\pi$ if desired. If $\boldsymbol{x}(t)$ is a solution of the $n$-body problem, then $\lambda\boldsymbol{x}(\lambda^{-3/2}t)$ is the solution as well.

## 3 Split eight

We search each trajectory as an extremal of the action functional

$$\mathcal{A}(q(t)) = \int_{t_1}^{t_2} L(\boldsymbol{r}, \dot{\boldsymbol{r}})\,\mathrm{d}t\,,$$

where $L$ is the Lagrangian of the problem $L = K + U$,

$$\begin{aligned}K &= \frac{1}{2}(m_1|\dot{\boldsymbol{r}}_1|^2 + m_2|\dot{\boldsymbol{r}}_2|^2 + m_3|\dot{\boldsymbol{r}}_3|^2),\\ U &= \frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}.\end{aligned}$$

We minimize the action functional $\mathcal{A}$ in the space of the $2\pi$-periodic function. So, we search the solution in the form

$$\begin{aligned}x_j(t) &= \sum_{i=1} C_{xi}^j \cos it + S_{xi}^j \sin it,\\ y_j(t) &= \sum_{i=1} C_{y_i}^j \cos it + S_{y_i}^j \sin it,\end{aligned}$$

where $j$ is the body number. We consider the inertial space; so, we can search the functions $x_j$, $y_j$ for two bodies only.

Thus, we have the following nonlinear programming problem:

$$\begin{aligned}&\min A(C_{xi}^j, C_{y_i}^j, C_{xi}^j, C_{y_i}^j), \quad i = 1,\ldots,n, \quad j = 1,2,\\ &g_k(C_{xi}^j, C_{y_i}^j, C_{xi}^j, C_{y_i}^j) = 0, \quad k = 1,\ldots,m,\\ &g_l(C_{xi}^j, C_{y_i}^j, C_{xi}^j, C_{y_i}^j) \le 0, \quad l = m+1,\ldots,l.\end{aligned}$$

Additionally, we have the extra relations $g_k$ if we search for a solution with some symmetry.

Methods of the nonlinear programming guarantee of finding the solution if the action functional $\mathcal{A}$ and the constraints $g_i$ are convex functions. For the $n$-body problem, it is not the case. However, these methods often work in the non-convex cases. To determine the coefficients, we use the following two nonlinear optimization systems: LOQO by

 

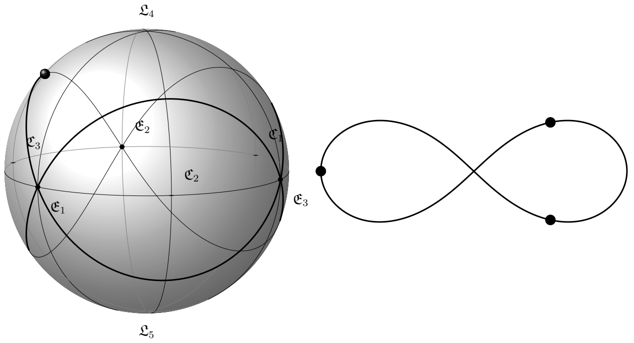

**Fig. 1** Figure-eight trajectory on the shape sphere (*left*), black ball is in the initial configuration, the Euler points: $\mathfrak{E}_i$, the collision points: $\mathfrak{C}_i$; and on the Euclidean plane (*right*), three points are in the initial configuration.

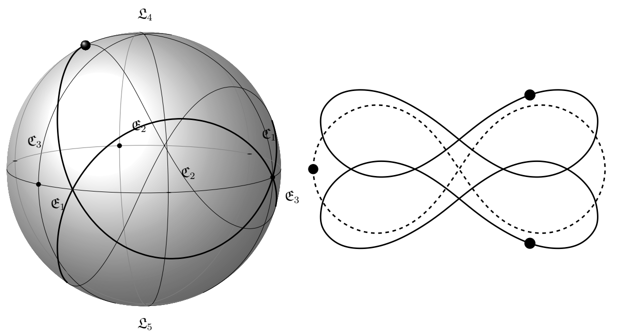

**Fig. 2** Split figure-eight trajectory on shape sphere (*left*), black ball is in the initial configuration, and on the Euclidean plane, three points in the initial configuration, the body of mass 0.97 moves along the dashed curve, bodies of mass 1 move along the symmetric solid line.

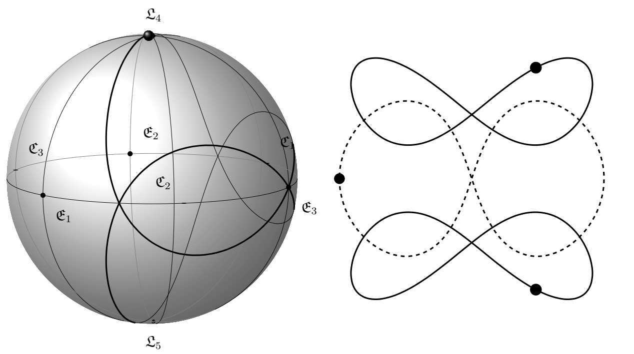

**Fig. 3** Split figure-eight trajectory on the shape sphere (*left*), black ball is in the initial configuration close to the Lagrange point $\mathfrak{L}_4$; on the Euclidean plane, three points are in the initial configuration, the body of mass 0.87 moves along the dashed curve, bodies of mass 1 move along the symmetric solid line.

Vanderbei (2007) and KNITRO by Byrd, Nocedal & Waltz (2006). Both systems give similar results.

For the figure-eight trajectory, we have a simple choreography and symmetry around the $x$- and $y$-axis.

$$
\begin{aligned}
x(t) &= \sum_{\substack{i=1\\ i\neq 3k}}^{N} C_{2i-1}\cos(2i-1)\,t + S_{2i-1}\,\sin(2i-1)\,t,\\
y(t) &= \sum_{\substack{i=1\\ i\neq 3k}}^{N} C_{2i}\,\cos 2i\,t + S_{2i}\,\sin 2i\,t,\\
x_j(t) &= x\left(t+2\pi(j-1)/3\right),\\
y_j(t) &= y\left(t+2\pi(j-1)/3\right), \quad j=1,2,3.
\end{aligned}
$$

The $2(N-N/3)$ coefficients for the expansions are truncated to $N$ harmonics. If we limit $N$ by 24, the amplitude of the neglected coefficients is less than $5\times10^{-7}$.

In Fig. 1, the next series are used (up to $10^{-3}$):

$$
\begin{aligned}
x =\; & 0.548\cos(t) + 0.949\sin(t) - 0.013\cos(5t)\\
& + 0.022\sin(5t) + 0.003\cos(7t) + 0.005\sin(7t),\\
y =\; & 0.292\cos(2t) + 0.169\sin(2t) + 0.048\cos(4t)\\
& - 0.028\sin(4t) + 0.003\cos(8t) + 0.002\sin(8t)\\
& + 0.001\cos(10t).
\end{aligned}
$$

In the left side of Fig. 1, the eight-trajectory is drawn on the shape sphere. The black ball at the left of the north hemisphere is a point corresponding to the initial configuration of these three bodies (the isosceles triangle). At the right side, the figure-eight trajectory is shown on Euclidean plane. The Euler points are $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, and the collision points are $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$.

Make the symmetry weaker. Now the trajectory is not the simple choreography, and each body moves on its own trajectory. If one supposes $m_1 = m_2$, the trajectories have the form

$$
\begin{aligned}
x_1(t) &= \sum_{i=1}^{N} C_{2i-1}\cos(2i-1)\,t + S_{2i-1}\,\sin(2i-1)\,t,\\
y_1(t) &= \quad b_0 + \sum_{i=1}^{N} C_{2i}\,\cos 2it + S_{2i}\,\sin 2it,\\
x_2(t) &= \sum_{i=1}^{N} C_{2i-1}\cos(2i-1)\,t - S_{2i-1}\,\sin(2i-1)\,t,\\
y_2(t) &= -b_0 - \sum_{i=1}^{N} C_{2i}\,\cos 2i\,t + S_{2i}\,\sin 2i\,t,\\
x_3(t) &= -\big(m_1x_1(t) + m_2x_2(t)\big)/m_3,\\
y_3(t) &= -\big(m_1y_1(t) + m_2y_2(t)\big)/m_3.
\end{aligned}
$$

We have to determine $2N+1$ coefficients.

For $m_1 = m_2 = 1$, $m_3 = 0.97$, we obtain

$$
\begin{aligned}
x_1 =\; & 0.526\cos(t) + 0.883\sin(t) + 0.006\cos(3t)\\
& -0.060\sin(3t) - 0.022\cos(5t) + 0.014\sin(5t)\\
& +0.008\cos(7t) + 0.003\sin(7t) - 0.004\sin(9t)\\
& -0.001\cos(11t) - 0.001\sin(11t) + 0.001\sin(13t),\\
y_1 =\; & 0.311\\
& +0.213\cos(2t) + 0.206\sin(2t) + 0.031\cos(4t)\\
& -0.043\sin(4t) - 0.017\cos(6t) + 0.001\sin(6t)\\
& +0.004\cos(8t) + 0.005\sin(8t) - 0.001\cos(10t)\\
& -0.002\sin(10t) - 0.001\cos(12t).
\end{aligned}
$$

In Fig. 2, the split eight-trajectory with $m_1 : m_2 : m_3 = 1 : 1 : 0.97$ is shown on the shape sphere and on the Euclidean plane. On the Euclidean plane, the body of mass 0.97 moves along the dashed curve, two other masses move along the two symmetric solid curves.

Let us change the ratio of the bodies' masses. Now, $m_1 : m_2 : m_3 = 1 : 1 : 0.87$.

$$
\begin{aligned}
x_1 = \; & 0.444\cos(t) + 0.720\sin(t) + 0.018\cos(3t) \\
& -0.092\sin(3t) - 0.032\cos(5t) + 0.014\sin(5t) \\
& +0.014\cos(7t) + 0.008\sin(7t) - 0.001\cos(9t) \\
& -0.008\sin(9t) - 0.003\cos(11t) + 0.003\sin(11t) \\
& +0.002\cos(13t) + 0.001\sin(13t) - 0.001\sin(15t) \\
& -0.001\cos(17t) + 0.001\sin(17t) + 0.001\cos(19t), \\
y_1 = \; & 0.612 \\
& +0.139\cos(2t) + 0.225\sin(2t) + 0.033\cos(4t) \\
& -0.054\sin(4t) - 0.026\cos(6t) + 0.007\cos(8t) \\
& +0.010\sin(8t) + 0.002\cos(10t) - 0.006\sin(10t) \\
& -0.003\cos(12t) + 0.001\sin(12t) + 0.002\cos(14t) \\
& +0.001\sin(14t) - 0.001\sin(16t) - 0.001\cos(18t).
\end{aligned}
$$

This trajectory is shown in Fig. 3. On the shape sphere the initial point is close to the north pole $\mathfrak{L}_4$. The trajectory of the smallest body (right) is drawn by a dashed line. The two other bodies move along the curves that do not intersect each other and lie in different semiplanes.

Two points, $m_1$ and $m_2$, move in different semiplanes if the mass ratio is less then 0.96. Their trajectories do not intersect each another. The initial configuration for the mass ratio 0.87 is very close to the equilateral triangle. Montgomery proved (see Theorem 2 and Corollary 1 in Montgomery 2002) that the height function $z(t)$ for the zero angular momentum solution has exactly one critical point between any two consecutive zeros. Moreover, $z = \pm 1$ (or $\theta = \pm\pi/2$) if the configuration is Lagrangian. In Fig. 3, the trajectory on the shape sphere passes very close to $L_4$, so, this type trajectory does not exist for the mass ratio less than 0.86.

## 4 Free-fall orbits

Here, we consider the free-fall three-body problem. This is the three-body problem with zero initial velocities; thus, the angular momentum equals to zero as well.

The first research on the free-fall three-body problem is dated back to researches by Meissel and Burrau who studied the Pythagorean problem: Three points with masses 3, 4, and 5 are located at vertices of the Pythagorean triangle and have zero initial velocity. Meissel expected that these initial condition would yield a periodic orbit, but he did not succeed. Burrau picked up the integration method and published results in 1913. More than fifty years later, Szebehely & Peters (1967) integrated the problem much further. And after a series of close encounters, the points go to infinity, although at some intermediate instant the configuration of points and velocity are close to the initial one. Slightly varying sides, Standish (1970) found periodic orbits close to the Pythagorean one.

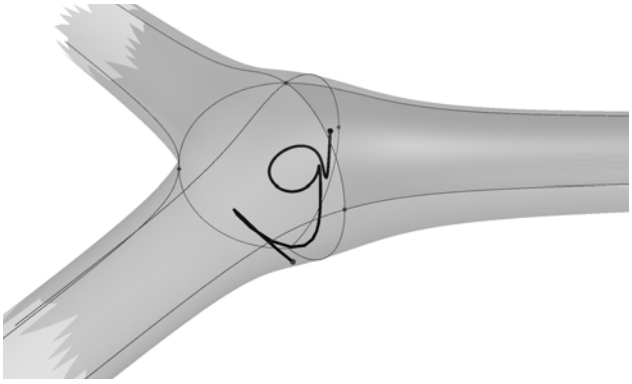

**Fig. 4** Hill surface for the three body problem in the shape space, $m_1 = m_2 = m_3 = 1$. Horizontal lines correspond to the degenerated triangles, other lines correspond to the isosceles triangles; the solid black line shows the trajectory of the free-fall solution; so, 2 boundary points (black) lie on the Hill surface.

Now consider the Hill surface $U = h$. Note that in 1988, Moeckel studied the qualitative features of the three-body problem using the Hill surface in a reduced space (Moeckel 1988). The free-fall solutions have one point on the Hill surface. Suppose this happens at the instant $t = 0$. The problem of $n$ bodies is reversible: if $\boldsymbol{x}(t)$ is a solution, then $\boldsymbol{x}(-t)$ is a solution as well. Once a periodic solution of the free-fall three-body problem begins on the Hill surface, this solution should reach the other point on the Hill surface in half of the period and then come back along the same path to the initial position. Such orbits are called the Brake ones. Without loss of generality, we can search the initial conditions for a periodic solution on the Hill surface; and due to the scale symmetry, we can set $h = -1$. Such a surface in the shape space is shown in Fig. 4.

Here, we deal with objects of equal masses $m_1 = m_2 = m_3 = 1$. Consider their trajectories in the shape space with coordinates $r$, $\varphi$, and $\theta$,

$$
\begin{aligned}
x &= r\cos\varphi\cos\theta, \\
y &= r\sin\varphi\cos\theta, \\
z &= r\sin\theta,
\end{aligned}
$$

then

$$r_{ij}^2 = r^2[1 - \cos\theta\,\cos(\varphi - \varphi_k)].$$

With fixed energy $h = -1$, the Hill surface is described as

$$r(\varphi, \theta) = \sum_{k=1}^{3} \frac{1}{\sqrt{1 - \cos\theta\,\cos(\varphi - \varphi_k)}}.$$

We scan our Hill surface and, looking for the trajectory, attain the Hill surface once more. If so, this is a periodic orbit. We can only test $\varphi \in (0, \pi/3)$, $\theta \in (0, \pi/2)$. The numerical integration using the Waldvogel regularization (Waldvogel 1972) yields the periodic free-fall orbit

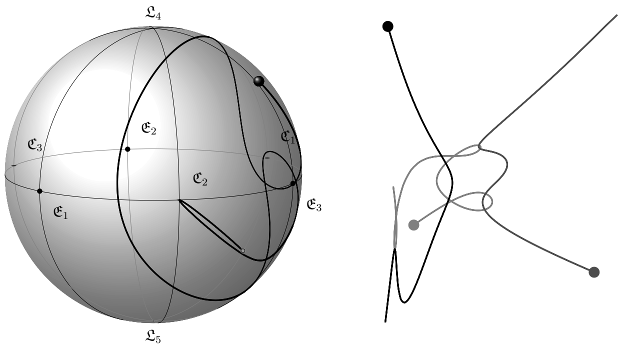


**Fig. 5** Free-fall three-body problem orbit on the shape sphere (*left*), the black ball is in the initial configuration, $m_1 = m_2 = m_3 = 1$, $h = -1$; the initial point and point (dark gray) in $T/2$ lie on the Hill surface.

with

$$\varphi_0 = 0.070800, \quad \theta_0 = 0.739568,$$

$$x_1 = 0.4842574 \quad x_2 = \quad 0.4842574 \quad x_3 = -0.9685148,$$
$$y_1 = 2.1991441 \quad y_2 = -2.1240441 \quad y_3 = -0.0751000.$$

$$r_{1_0} : r_{2_0} : r_{3_0} = 0.58099 : 0.624228 : 1\,.$$

The second "conjugate" point on the Hill surface is

$$\varphi_h = -0.489050, \quad \theta_h = -0.421551,$$
$$r_{1_h} : r_{2_h} : r_{3_h} = 2.070005 : 2.826337 : 1\,.$$

The period is $T = 20.00988$, the minimal distance between bodies $r_{\rm min} = 0.00136$. In Fig. 4, the solution is drawn as the thick solid curve. Black points lie on the Hill surface. All points of the solution, except two boundary ones, lie inside the Hill surface.

In Fig. 5, this trajectory is shown on the shape sphere (left) and in the Euclidean plane. The black ball is the first (initial) point on the shape sphere, the small gray circle is the second ("conjugate") one.

## 5 Conclusion

Two classes of orbits with vanishing angular momentum are investigated. The first one is an extension of the classic figure-eight orbit, i.e., the split figure-eight one. Cases are considered where two masses are equal and the third one is slightly smaller. Varying the value of the third mass, we obtained symmetric periodic orbits. Starting with $m_1 : m_2 : m_3 = 1 : 1 : 0.96$, the two quasi-eights trajectories do not intersect each other and lie in two different semiplanes.

The isosceles configuration is close to the equilateral triangle, and the trajectories are close to $\mathfrak{L}_4$. Since the height of the trajectory is a monotonic function between two sequential extrema, such trajectories with mass distribution less than $m_1 : m_2 : m_3 = 1 : 1 : 0.86\ldots$ do not exist.

Consideration of the three body problem in a reduced space (the shape space) allow us to reduce the domain of the initial condition for the free-fall problem to a two dimensional fragment of $R^2$: $\varphi \in (0, \pi/3)$, $\theta \in (0, \pi/2)$.

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## References

Barutello, V., Ferrario D., & Terracini S. 2008, Archive for Rational Mechanics & Analysis, 190, 189
Broucke, R., & Lass, H. 1973, Celest. Mech., 8, 5
Broucke R., & Boggs D. 1975, Celest. Mech., 11, 13
Burrau, C. 1913, Astron. Nachr., 195, 6
Byrd, R. H., Nocedal, J., & Waltz, R.A. 2006, in Large-Scale Nonlinear Optimization, eds. G. di Pillo, & M. Roma, Springer, 35
Chenciner A., & Montgomery R. 2000, Ann. of Math, 152, 881
Henon M. 1976, Celest. Mech., 13, 267
Hsiang Wu.-Yi., & Straume E. 1994, preprint, CPAM-620, Center for Pure and Applied Math, Berkeley
Lemaitre G. 1955, Vistas Astron. 1, 207
Moeckel R. 1988, Contemp. Math. 81, 1
Moeckel R., & Montgomery R. 2013, Pacific J. Math., 262, 1
Montgomery R. 1995, Nonlinearity, 9, 1341
Montgomery R. 2002, Archive for Rational Mechanics and Analysis, 162, 311
Moore C. 1993, Phys. Rev. Lett., 70, 3675
Schubart J 1956, Astron. Nachr., 283, 17.
Suvakov M., & Dmitrasinovich V. 2013, Phys. Rev. Lett., 110, 114301
Standish E. M. Jr. 1970, in Periodic Orbits, Stability and Resonance, ed. G.E.O. Giacaglia, 375
Szebehely, V., & Peters, C. F. 1967, AJ, 72, 876
Titov V. 2006, in Few-Body Problem: Theory and Computer Simulation, ed. C. Flynn, Annales Universitatis Turkuensis, Astronomica-Chemica-Physica-Mathematica, 358, 9
Vanderbei, R.J. 2007, Linear Programming: Foundations and Extensions, Kluwer Academic Publishers
Waldvogel J. 1972, Celest. Mech, 6, 221

 