The quantum relativistic two-body bound state. I. The spectrum

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In the framework of a manifestly covariant quantum theory on space-time, it is shown that the ground state mass of a relativistic two-body system with O(3,1) symmetric potential is lower when represented by a wave function with support in an O(2,1) invariant subspace of the spacelike region. The wave functions for the relativistic bound states are obtained explicitly. Coulomb type binding, the harmonic oscillator, and the relativistic square well are treated as examples. The mass spectrum is determined by a differential equation in the invariant spacelike interval ρ, which can be put into correspondence with the radial part of a nonrelativistic Schrödinger equation with potential of the same form, where r is replaced by ρ. In the case that the binding is small compared to the particle masses, the mass spectrum (bounded below) is well-approximated by the results of the nonrelativistic theory. The eigenfunctions transform under the full Lorentz group as elements of an induced representation with O(2,1) little group. This representation is studied in a succeeding paper.

I. INTRODUCTION

In nonrelativistic quantum mechanics, the use of Schrödinger's time-independent equation with central potentials for the study of bound states has been very successful in the description of atomic spectra and in the construction of wave functions as a basis for perturbation theory for the treatment of non-spherically symmetric interactions and radiation. A corresponding relativistic theory, with O(3,1) symmetric direct action potentials, could be expected to offer analogous applications, with the advantage of maintaining covariance, essential for consistency in the determination of mass spectra and for its application to radiation theory.[1] Such a theory should include the nonrelativistic results when the binding is small compared to the particle masses, and provide bounds for the applicability of the nonrelativistic theory.

In this paper, we shall study the bound state problem in the framework of a manifestly covariant quantum theory[2,3] that treats *events* (the occurrence of physical phenomena locally at space-time points), rather than *particles* (the occurrence of physical phenomena with functional dependence along world lines), as the fundamental physical entities.[4]

The construction of a manifestly covariant mechanics, both classical and quantum, of the type that we shall use, was carried out by Stueckelberg in 1941,[2] for the case of a single particle in an external field. He considered the phenomena of pair annihilation and creation as a manifestation of the development, in each case, of a single world line that curves in such a way that in one half-space of time the line passes twice, and in the other, not at all. To describe such a curve, parametrization by the variable t is ineffective, since the trajectory is not single valued. He therefore introduced a parametric description, with parameter τ along the world line. Hence one branch of the curve is generated by motion in the positive sense of t as a function of increasing τ, and the other branch by motion in the negative sense of t. The second branch is identified with the antiparticle, a rule that also emerged in Feynman's quantum electrodynamics.[2]

The motion, in space-time, of the point generating the world line, which we shall call an *event* (and has properties of space-time position and energy momentum), is governed in the classical case by the Hamilton equations in space-time

$$\frac{dx^\mu}{d\tau}=\frac{\partial K}{\partial p_\mu},\quad \frac{dp^\mu}{d\tau}=-\frac{\partial K}{\partial x_\mu},\tag{1.1}$$

where $x^\mu=(t,\mathbf{x})$, $p^\mu=(E,\mathbf{p})$ [we take $c=1$ and $g_{\mu\nu}=(-1,1,1,1)$] and the evolution generator K is a function of the canonical variables x_μ, p_μ. For the special case of free motion,

$$K_0=p^\mu p_\mu/2M,\tag{1.2}$$

where M is an intrinsic parameter assigned to the generic event, and hence

$$\frac{dx^\mu}{d\tau}=\frac{p^\mu}{M}.\tag{1.3}$$

It then follows that

$$\frac{d\mathbf{x}}{dt}=\frac{\mathbf{p}}{E},\tag{1.4}$$

consistent with standard relativistic kinematics. We note, however, that the mass squared $m^2=-p^\mu p_\mu$ is a dynamical variable since $\mathbf{p}$ and E are considered to be kinematically independent, and therefore it is not taken to be equal to a given constant. The set of values taken by m^2 in a particular dynamical context is determined by initial conditions and the dynamical equations.

In the quantum theory, $\mathbf{x},t$ (and $\mathbf{p},E$) denote operators satisfying the commutation relations (we take $\hbar=1$)

$$[x^\mu,p^\nu]=ig^{\mu\nu}.\tag{1.5}$$

The state of a one-event system is described by a wave function $\psi_\tau(x)\in L^2(R^4)$, a complex Hilbert space with measure $d^4x=d^3x\,dt$ satisfying the equation[2]

$$i\frac{\partial\psi_\tau(x)}{\partial\tau}=K\psi_\tau(x).\tag{1.6}$$

This equation, designed to provide a manifestly covariant

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description of relativistic phenomena, is similar in form to the nonrelativistic Schrödinger equation. Although free motion is determined by the operator form of K_0 of Eq. (1.2), i.e., the d'Alembertian, which is hyperbolic ($p_\mu p^\mu \equiv -\partial_\mu \partial^\mu$ instead of the elliptic operator $\mathbf{p}^2 \equiv -\nabla^2$), the same methods may be used for studying Eq. (1.6) as for the nonrelativistic Schrödinger equation.

The unperturbed evolution of a free event is described by a wave packet of the form

$$\psi_\tau(x) = \int d^4p f(p) \exp\left\{ -i\left(\frac{p^2}{2M}\right)\tau\right\} e^{ip\cdot x}, \qquad (1.7)$$

where $p^2 = p^\mu p_\mu$, $p\cdot x = p^\mu x_\mu$. The stationary phase contribution to $\psi_\tau(x)$ (Ehrenfest motion) is at the point

$$x^\mu{}_c \simeq (p^\mu{}_c/M)\tau, \qquad (1.8)$$

where $p^\mu{}_c$ is the peak value in the distribution $f(p)$. In the case where $p^0{}_c = E_c < 0$, we see, as in Stueckelberg's classical example, that

$$\frac{dt_c}{d\tau} \simeq \frac{E_c}{M} < 0. \qquad (1.9)$$

It has been shown[4] in the analysis of an evolution operator with minimal electromagnetic interaction, of the form

$$K = (p - eA(x))^2/2M, \qquad (1.10)$$

that the CPT conjugate wave function is given by

$$\psi^{\mathrm{CPT}}{}_\tau(\mathbf{x},t) = \psi_\tau(-\mathbf{x},-t), \qquad (1.11)$$

with $e \to -e$. For the free wave packet, one has

$$\psi^{\mathrm{CPT}}{}_\tau(\mathbf{x},t) = \int d^4p f(p) \exp\left\{ -i\left(\frac{p^2}{2M}\right)\tau\right\} e^{-ip\cdot x}. \qquad (1.12)$$

The Ehrenfest motion in this case is

$$x^\mu{}_c \simeq -(p^\mu{}_c/M)\tau; \qquad (1.13)$$

if $E_c < 0$, we see that the motion of the event in the CPT conjugate state is in the positive direction of time, i.e.,

$$\frac{dt_c}{d\tau} \simeq -\frac{E_c}{M} = +\frac{|E_c|}{M}, \qquad (1.14)$$

and one obtains the representation of a positive energy generic event with the opposite sign of charge, i.e., the antiparticle.[5]

Equation (1.6), with K of the form (1.10), leads to the conservation law

$$\frac{\partial \rho}{\partial \tau} = -\partial_\mu j^\mu(x), \qquad (1.15)$$

where

$$\rho(x) = |\psi_\tau(x)|^2 \qquad (1.16)$$

and

$$j^\mu(x) = -(ie/2M)\{\psi^*{}_\tau(x)(\partial^\mu - ieA^\mu(x))\psi_\tau(x) - ((\partial^\mu + ieA^\mu(x))\psi^*{}_\tau(x))\psi_\tau(x)\}. \qquad (1.17)$$

It is clear from (1.15) that $j^\mu(x)$ cannot be the source of a Maxwell field since

$$\partial_\nu F^{\mu\nu}(x) = J^\mu(x) \qquad (1.18)$$

implies that

$$\partial_\mu J^\mu(x) = 0. \qquad (1.19)$$

As observed by Stueckelberg, who gave a geometrical argument in his 1942 paper[2] (or by application of the Riemann–Lebesgue lemma[3]) $\rho_\tau(x) \to 0$ as $\tau \to \pm\infty$, and hence, for asymptotically free motion,[4,6]

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau\, j^\mu{}_\tau(x). \qquad (1.20)$$

Since particles are observed in the laboratory, directly or indirectly, by means of electromagnetic interaction, we see that the notion of a *particle* is associated with the entire world line, i.e., the *set of events* generated by the motion over all τ. We have called this construction, of an object that has the properties of a particle, from a set of events constituting the world line, "concatenation."[4]

For the treatment of systems of more than one event (generating world lines of more than one particle), one assumes the unperturbed evolution generator to be of the form[3]

$$K_0 = \sum_{i=1}^{N} \frac{p_i{}^2}{2M_i}. \qquad (1.21)$$

In the presence of electromagnetic interaction (for spinless particles) one uses the minimal coupling form, which is a generalization of (1.10),

$$K = \sum_{i=1}^{N} \frac{(p_i - e_i A(x_i))^2}{2M_i}. \qquad (1.22)$$

As pointed out above, there is a class of model systems, for which solutions can be achieved using straightforward methods, which involve only effective action-at-a-distance (direct action) potentials, where the evolution generator is of the form

$$K = \sum_{i=1}^{N} \frac{p_i{}^2}{2M_i} + V(x_1, x_2, \ldots, x_N). \qquad (1.23)$$

Note that in this case the potential function enters into the dynamical evolution equation as a term added to the generator of the free motion, and therefore corresponds to a spacetime coordinate-dependent interaction mass.

Equations (1.1) become

$$\frac{dx_i{}^\mu}{d\tau} = \frac{\partial K}{\partial p_{i\mu}}, \quad \frac{dp_i{}^\mu}{d\tau} = -\frac{\partial K}{\partial x_{i\mu}}. \qquad (1.24)$$

The program is to solve the dynamical equation (1.6) with the dynamical evolution operator (1.22) or (1.23) [or Eqs. (1.24) for the classical case] governing the motion of events in interaction with each other and with external fields; predictions of observable phenomena are then obtained *a posteriori* by concatenation of the historical sequence of events. We shall concentrate on the direct action form (1.23) in this paper in our treatment of two-body bound states. As we shall see, the *relative* motion of bound states is represented by τ-independent wave functions (up to a phase). The center of mass (since the evolution generator is quadratic in energy momentum, one may always carry out a separation of variables for the center of mass motion) evolves as a free event, however, and concatenation then provides a world history of the two-body bound state that consists of a straight world line for the (Ehrenfest motion of) the center of mass associated with a stationary distribution for the relative motion.

Nonrelativistic Schrödinger potential theory implicitly synchronizes points on the particle trajectories by assuming that interaction occurs between them at equal times, i.e., in the potential $V(|\mathbf{r}_1-\mathbf{r}_2|^2)$, where $\mathbf{r}_1$, the position of the first particle, and $\mathbf{r}_2$, of the second, are to be taken as positions on the trajectories at the same time t. This synchronization cannot be maintained in a relativistic framework. The synchronization of space-time *events*, corresponding to points along the particle world lines, can, nevertheless, be consistently and covariantly maintained by means of the universal evolution parameter τ. The two-body potential function, which we choose for Poincaré invariance to be of the form $V(\rho^2)$, where

$$\rho=\sqrt{(x_2{}^\mu-x_2{}^\mu)(x_{1\mu}-x_{2\mu})}\equiv\sqrt{(x_1-x_2)^2},$$

carries the implication that the events $x_1{}^\mu$ and $x_2{}^\mu$ interact at equal τ, and hence implies the existence of a synchronization of events.[7]

There have been many attempts to deal with the relativistic bound state problem. The Bethe–Salpeter method[8] and related techniques,[9] based on structures provided by quantum field theory, have been successful in describing spectra to high precision.[10] The quantum mechanical interpretation of the wave function in these approaches is, however, not completely clear.

Constraint Hamiltonian dynamics, introduced by Dirac,[11] for dealing with singular Lagrangians of the type arising in gauge theories, has been developed for relativistic mechanics in both the classical and quantum cases.[12] The form of the interaction potentials, however, which must be used in this approach, is highly restricted by the integrability conditions; possible forms for more than two particles are difficult to construct, and are not known in general.[13]

One of the advantages of the constraint formalism is that, in scattering processes, the asymptotic expectation value of $p_i{}^2$ for each of the particles is ensured to be the correct "on shell" value.[12] In the unconstrained form of mechanics that we shall use, there is no restriction on the structure of the potential function (other than the requirement that the resulting differential equations are mathematically well-defined) for any number of particles. The asymptotic behavior of the expectation value of $p_i{}^2$ for each particle in a scattering process (or in ionization from a bound state) is related to the asymptotic synchronization of events in the universal historical time τ.[14] Transitions, such as between μ and e masses, are admitted in this framework.

Some authors have discussed the relativistic two-body bound state in a framework similar to the one we use here.[15] In these works, it was assumed that the relative motion is free to penetrate the entire spacelike region. We shall show that, for the O(3,1) symmetric Coulomb-type potential, the ground state wave function with support in an O(2,1) invariant subregion of the full spacelike region has a lower mass eigenvalue than the ground state wave function with support in the full spacelike region. This phenomenon corresponds to a spontaneous breakdown of the O(3,1) symmetry of the differential equations.

The support of the wave function determines the range of synchronization of the two-event system, and our computation of excited states assumes that this synchronization is characteristic of the bound states and persists, i.e., their support also lies in the O(2,1) invariant subregion. The resulting mass spectrum, for the case in which the binding is small compared to the mass of the particles (as, for example, in atomic physics), essentially coincides with the nonrelativistic Schrödinger energy spectrum for the corresponding $V(r^2)$, for arbitrary $V(\rho^2)$. The method used here is applicable as well to the problem of the strong binding of light particles, such as light quarks in a hadron. If, however, the binding exceeds a critical strength (in case there *is* an ionization point), we find that the simple notion of a bound state as a composite of two systems with intrinsic properties determined asymptotically above the ionization point is untenable. Techniques will be presented elsewhere to take into account the effects of spin.[16]

Since the support of the bound state wave functions lies in a restricted O(2,1) invariant sector of the full spacelike region, the representations they provide for the full O(3,1) space-time symmetry must be of induced type [it is shown in the Appendix that an O(2,1) ladder cannot be constructed in the Hilbert space]. Under Lorentz transformations, the (unit) spacelike vector n_μ for which O(2,1) is the stabilizer subgroup transforms through all spacelike directions and covers the complete single sheeted unit hyperboloid. Under such transformations, the wave functions undergo an action of the O(2,1) little group, and are modified along orbits parametrized by this unit vector.

The induced representation is constructed as a family of Hilbert spaces with measure spaces restricted to a family of O(2,1) invariant sectors. The parameter n_μ appears, in this respect, to play the role of a continuous superselection rule.[17] In a sequel to this paper,[18] to be called II, the representations of O(3,1) obtained in this way are studied by classifying states according to the eigenvalues of the operators generating an O(3) subgroup of O(3,1). It is shown there that these constitute the canonical representations of Gel'fand of the principal series; they are unitary in the larger Hilbert space in which all of the Hilbert spaces labeled by n_μ are embedded with measure $d^4n\,\delta(n^2-1)$.

In Sec. II, we formulate the problem of reduced motion in an O(3,1) symmetric potential, and obtain the eigenvalue equation for the relative mass spectrum as a radial equation of Schrödinger type, with invariant ρ as the "radial" coordinate, and the O(3,1) Casimir operator $\frac{1}{2}M_{\mu\nu}M^{\mu\nu}$ as the coefficient of the "centrifugal" term. In Sec. III, the differential equations after separation of variables are obtained for a parametrization in terms of two angles θ, ϕ, and a hyperbolic angle β which, along with ρ, cover what we shall call the [O(2,1) invariant] restricted Minkowski space (RMS), a region obtained as the exterior of two hyperplanes tangent to the light cone and oriented along the z axis. This region may be visualized by folding the x, y coordinates together; in the resulting three-dimensional space, these hyperplanes become planes and intersect along the z axis (Fig. 1). Alterna-

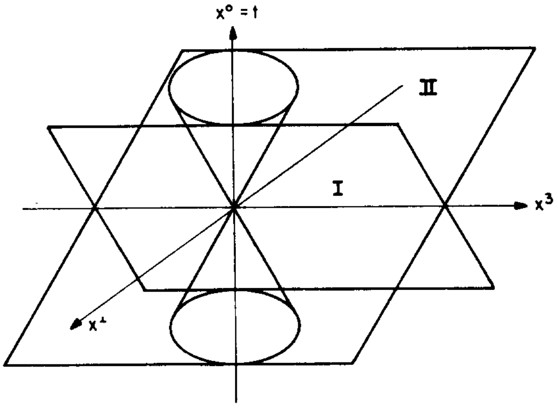

FIG. 1. The restricted Minkowski space (RMS) taken for the support of solutions of the eigenvalue equation in relative variables is designated as I, the region exterior to the two planes tangent to the light cone and intersecting along the x_3 axis ($\theta = 0,\pi$). The spatial coordinates x_1 and x_2 are folded into a single axis in this figure ($x_\perp$); in 3 + 1 dimensions the RMS is connected (but not simply connected, as seen from Fig. 2).

tively, we display this region in a projective space (Fig. 2).[19] The order of separation is first in ϕ, the azimuthal angle around the z axis, then in the O(2,1) boost parameter β to obtain the eigenvalue for the O(2,1) Casimir operator (the bound state levels are degenerate with respect to this quantum number). The separation equation for the remaining angle θ corresponds to the eigenvalue equation of the O(3,1) Casimir operator. The solutions and normalization conditions for these eigenvalue equations are given in Sec. IV. The separated equations for both θ and β variables have solutions that are associated Legendre functions, with "magnetic quantum number" determined by the O(2,1) Casimir. The separation function of θ has order determined by the O(3,1) Casimir. A geometrical interpretation is given in this section relating these quantum numbers to the usual nonrelativistic magnetic and orbital quantum numbers. In the nonrelativistic limit, these functions survive intact to play the usual role of the Legendre functions in the description of the nonrelativistic bound states.

In Sec. V the radial equation and invariant relative mass spectrum is discussed, and, in Sec. VI, we treat the examples of an O(3,1) invariant Coulomb-type potential (which reduces to the ordinary Coulomb potential in the nonrelativistic limit), the relativistic oscillator (where we find that no subsidiary conditions are required), and an O(3,1) invariant version of a square well. The lowest-order relativistic corrections to the corresponding nonrelativistic results are given in case the binding is small compared to the particle masses. For very large binding, exceeding a critical strength, we show that the simple idea of a bound state as a composite of two systems with intrinsic properties determined asymptotically above the ionization point (in case, as in the first and third examples, there is an ionization point) becomes untenable.

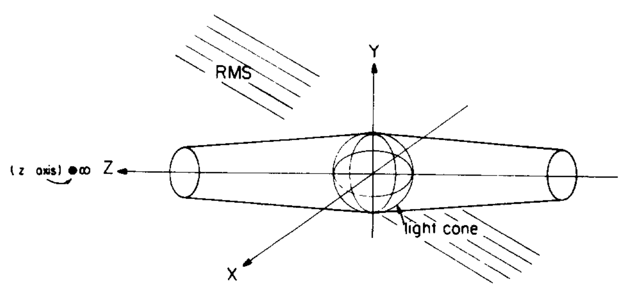

FIG. 2. The RMS in the projective space $\mathbf{R} = \mathbf{r}/t$; the unit sphere corresponds to the light cone. Each point corresponds to a line in Minkowski space. The point at ∞ along the Z axis is the z axis, and the point at $\mathbf{R} = 0$ is the t axis. The RMS is outside the cylinder $X^2 + Y^2 = 1$, i.e., $x^2 + y^2 \geqslant t^2$.

II. O(3,1) SYMMETRIC EQUATION OF MOTION AND THE EIGENVALUE EQUATION FOR REDUCED MOTION

We shall study in this section the evolution equation,[3]

$$i\frac{\partial}{\partial\tau}\Psi_\tau(x_1,x_2) = K\ \Psi_\tau(x_1,x_2), \tag{2.1}$$

where ($p_i{}^2 = p_i{}^\mu p_{i\mu} \equiv -\partial_i{}^\mu \partial_{i\mu}$),

$$K = p_1{}^2/2M_1 + p_2{}^2/2M_2 + V, \tag{2.2}$$

and $\Psi_\tau \in L^2(R^8)$.

We shall take the direct action potential V to have the O(3,1) symmetric form

$$V = V(\rho^2), \tag{2.3}$$

where

$$\rho^2 = (x_1 - x_2)^2 = (x_1 - x_2)^\mu (x_1 - x_2)_\mu . \tag{2.4}$$

We now separate the center of mass motion by defining the relative and center of mass variables with the natural choice[20]

$$P^\mu = p_1{}^\mu + p_2{}^\mu, \quad X^\mu = \frac{M_1 x_1{}^\mu + M_2 x_2{}^\mu}{M_1 + M_2}, \tag{2.5}$$

$$p^\mu = \frac{M_2 p_1{}^\mu - M_1 p_2{}^\mu}{M_1 + M_2}, \quad x^\mu = x_1{}^\mu - x_2{}^\mu, \tag{2.6}$$

where $m = M_1M_2/(M_1+M_2)$ and $M = M_1 + M_2$. Equation (2.1) can be represented as a direct integral over Hilbert spaces $L^2(R^4)$, with measure d^4x, labeled by values of the absolutely conserved P^μ. One obtains the family of equations

$$i\frac{\partial}{\partial\tau}\Psi_{P'\tau}(x) = \left[\frac{P'^2}{2M} + K_{\rm rel}\right]\Psi_{P'\tau}(x). \tag{2.7}$$

In this way, we have separated out the center of mass motion. The operator K has, in general, continuous spectrum, but on the Hilbert spaces that are elements of the direct sum, i.e., for each value P'^μ, $K_{\rm rel}$ may have discrete or continuous spectrum. This spectrum corresponds to the contribution of the relative motion to the mass spectrum; we shall call it the "mass spectrum of the relative motion." We shall study the discrete spectrum of this operator, and the corresponding eigenstates.

For the discrete spectrum, we write

$$\Psi_{P'\tau}(x) = \exp(-i(P'^2/2M)\tau) e^{-iK_a\tau}\psi_{P'}{}^{(a)}(x); \tag{2.8}$$

Eq. (2.7) then becomes (we suppress reference to P' in the following)

$$K_a\psi^{(a)}(x) = (-(1/2m)\partial_\mu\partial^\mu + V(\rho^2))\psi^{(a)}(x). \tag{2.9}$$

The (invariant) relative radial coordinate can be separated from the angular and hyperbolic angular variables in the d'Alembertian with the help of the O(3,1) Casimir operator,

$$\Lambda = \tfrac{1}{2} M_{\mu\nu} M^{\mu\nu}, \tag{2.10}$$

where

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \tag{2.11}$$

With the help of the commutation relations (1.5) [valid also for the relative coordinates defined by (2.5)], one obtains

$$\Lambda = x^2 p^2 + 2ix\cdot p - (x\cdot p)^2, \tag{2.12}$$

where

$$x\cdot p \equiv x^\mu p_\mu = -i\rho \frac{\partial}{\partial\rho}, \tag{2.13}$$

valid for spacelike or timelike values of x^μ. We therefore see that ($\Box \equiv -\partial^\mu \partial_\mu, \rho^2 = x^\mu x_\mu$)

$$\Lambda = \rho^2 \Box + 3\rho \frac{\partial}{\partial\rho} + \rho^2 \frac{\partial^2}{\partial\rho^2}, \tag{2.14}$$

or

$$\Box = -\frac{\partial^2}{\partial\rho^2} - \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{\Lambda}{\rho^2}. \tag{2.15}$$

Note that $\rho^2 \geqslant 0$ in the spacelike region [in the timelike region, ρ should be replaced by $i\rho'$, where $\rho'^2 = -x^\mu x_\mu$; in terms of the new variable ρ', it appears that the expression for $\Box$ in (2.15) has changed sign].

It then follows that Eq. (2.9) can be written as

$$K_a \psi^{(a)}(x) = \left\{ \frac{1}{2m} \left[-\frac{\partial^2}{\partial\rho^2} - \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{\Lambda}{\rho^2} \right] + V(\rho^2) \right\} \psi^{(a)}(x). \tag{2.16}$$

III. SEPARATION OF VARIABLES

Further separation of variables depends on the choice of the sector of Minkowski space in which one studies the differential equation (2.9) and the corresponding parametrization of these sectors by hyperbolic angular (which we shall call hyperangular) and angular variables.[21] Each sector is associated with a spectrum determined by its structure and the boundary conditions applied to the solutions in that sector.

In addition to the more widely used decomposition of Minkowski space into the timelike and full spacelike regions, we shall use a further decomposition of the spacelike region into two subregions [invariant under an O(2,1) subgroup of O(3,1)]. One of these sectors (I) consists of the space-time points external (in spacelike directions) to two hyperplanes tangent to the light cone that are oriented along the z axis (the direction must be chosen to define this space). The second (II) consists of the space-time points in the sector interior (timelike direction) to these hyperplanes, but excluding the light cone. In Fig. 1 this decomposition is shown schematically by folding the two space axes x, y together (defining the coordinate $x_\perp$); in the resulting three-dimensional space, the two hyperplanes become planes and intersect along the z axis.

Alternatively, one may represent the light cone in a projective three-dimensional space[19] by dividing the equation $|\mathbf{r}|^2 - t^2 = 0$ by t^2 to obtain $|\mathbf{R}|^2 = 1$ ($\mathbf{R} = \mathbf{r}/t$), the equation for the unit sphere. The region I is characterized by $x^2 + y^2 - t^2 \geqslant 0$, translationally invariant in z. In the projective space, this region is mapped to $X^2 + Y^2 \geqslant 1$, the space exterior to the cylinder, parallel to the Z axis, which circumscribes the unit sphere. The space interior to the cylinder, excluding the unit sphere, corresponds to region II. We remark that the point at infinity on the Z axis ($z/t = \infty$) corresponds to the z axis, and the point at the center of the unit sphere ($\sqrt{x^2+y^2+z^2}/t = 0$) corresponds to the t axis. This representation is shown in Fig. 2.

The subgroup O(2,1) of O(3,1) leaving sectors I and II invariant has been used by Bargmann[22] as a little group for the construction of an induced representation of the Poincaré group with the direction of the z axis (momentum) providing the parameter along the orbit. In this construction, he used functions with support in the interior sector II. Zmuidzinas,[23] in his study of the unitary representations of the Lorentz group using differential equations, studied both the interior sector II and the exterior sector I. We shall see that solutions of Eq. (2.16) with support in the exterior sector I are associated with the physical bound states of the two-body problem with O(3,1) symmetric potential. We shall call this sector the restricted Minkowski space (RMS) oriented, as we have described it here, along the z axis.

The parametrization

$$\begin{aligned} x^0 &= \rho \sin\theta \sinh\beta, \quad x^1 = \rho \sin\theta \cos\phi \cosh\beta, \\ x^2 &= \rho \sin\theta \sin\phi \cosh\beta, \quad x^3 = \rho\cos\theta \end{aligned} \tag{3.1}$$

covers the RMS for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, $-\infty < \beta < \infty$, and $0 \leqslant \rho = \sqrt{|\mathbf{r}|^2 - t^2} < \infty$ (we shall use $\mathbf{x}$ and $\mathbf{r}$ interchangeably). We record, for comparison, the parametrization

$$\begin{aligned} x^0 &= \rho\sinh\beta, \quad x^1 = \rho\cosh\beta\cos\phi\sin\theta, \\ x^2 &= \rho\cosh\beta\sin\phi\sin\theta, \quad x^3 = \rho\cosh\beta\cos\theta, \end{aligned} \tag{3.2}$$

for the full spacelike region.

The properties of the wave functions and the spectrum of $K_{\rm rel}$ obtained in the full spacelike region[15] have important differences from those expected of physical bound states for spinless particles. In particular, separation of variables in the full spacelike region parametrized by (3.2) leads to degeneracy in $\mathbf{L}^2$ for *every* O(3,1) symmetric potential and the nonrelativistic limit of the spectrum obtained does not coincide with the nonrelativistic hydrogen spectrum in the case $V \propto 1/\rho$.

We therefore proceed to study Eq. (2.16) in the case that the wave functions have support in sector I, the RMS. Introducing the usual three-vector notation

$$L_i = \tfrac{1}{2}\epsilon_{ijk}(x^j p^k - x^k p^j), \tag{3.3}$$

$$A^i = x^0 p^i - x^i p^0, \tag{3.4}$$

for i,j,k running from 1 to 3, and ϵ_{ijk} the totally antisymmetric (unit) tensor in three dimensions, the nonvanishing O(3,1) Casimir operator (the second Casimir operator $\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} M_{\mu\nu} M_{\lambda\sigma} = A\cdot L$ is identically zero for the spinless case) is

$$\Lambda = \mathbf{L}^2 - \mathbf{A}^2. \tag{3.5}$$

In terms of the parameters of the RMS, the differential operators $\partial/\partial x^\mu$ are

$$\frac{\partial}{\partial x^0} = -\sin\theta\sinh\beta\frac{\partial}{\partial\rho} - \frac{1}{\rho}\cos\theta\sinh\beta\frac{\partial}{\partial\theta} + \frac{\cosh\beta}{\rho\sin\theta}\frac{\partial}{\partial\beta},$$

$$\frac{\partial}{\partial x^1} = \cos\phi\left(\sin\theta\cosh\beta\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\theta\cosh\beta\frac{\partial}{\partial\theta} - \frac{\sinh\beta}{\rho\sin\theta}\frac{\partial}{\partial\beta}\right) - \sin\phi\frac{1}{\rho\sin\theta\cosh\beta}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial x^2} = \sin\phi\left(\sin\theta\cosh\beta\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\theta\cosh\beta\frac{\partial}{\partial\theta} - \frac{\sinh\beta}{\rho\sin\theta}\frac{\partial}{\partial\beta}\right) + \cos\phi\frac{1}{\rho\sin\theta\cosh\beta}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial x^3} = \cos\theta\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\theta\frac{\partial}{\partial\theta}. \tag{3.6}$$

It then follows that

$$\Lambda = -\frac{\partial^2}{\partial\theta^2} - 2\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}N^2, \tag{3.7}$$

where

$$N^2 = L_3{}^2 - A_1{}^2 - A_2{}^2 \tag{3.8}$$

is the Casimir operator of the O(2,1) subgroup of O(3,1) leaving the z axis (and the RMS) invariant. In terms of the variables of sector I, this operator is given by

$$N^2 = \frac{\partial^2}{\partial\beta^2} + \tanh\beta\frac{\partial}{\partial\beta} - \frac{1}{\cosh^2\beta}\frac{\partial^2}{\partial\phi^2}. \tag{3.9}$$

We emphasize that these operators are not "restrictions," in the sense of projection, of the operators defined on functions with support on all of space-time, or on the full spacelike region. They are constructed as operators on functions with support in the RMS as their natural domain.

Since the operator Λ defined in (3.7) (and associated boundary conditions) is essentially different from the corresponding operator applicable to functions with support on the whole spacelike region, its spectrum is different as well.

The invariant measure in $L^2(R^4)$ on sector I of the Minkowski space is

$$d\mu = \rho^3\sin^2\theta\cosh\beta\, d\rho\, d\phi\, d\beta\, d\theta. \tag{3.10}$$

As a complete commuting set of dynamical variables, we use the subset of symmetric operators (we assume they are self-adjoint in the following and shall explicitly find their spectra),

$$\{K_{\rm rel}, L_3, N^2, \Lambda\}. \tag{3.11}$$

The generators of the O(2,1) subgroup, leaving the quadratic form $x_1{}^2 + x_2{}^2 - x_0{}^2$ invariant, are

$$H_\pm \equiv A_1 \pm iA_2 = e^{\pm i\phi}\left(-i\frac{\partial}{\partial\beta} \pm \tanh\beta\frac{\partial}{\partial\phi}\right) \tag{3.12}$$

and

$$L_3 = -i\frac{\partial}{\partial\phi}. \tag{3.13}$$

The remaining generators of O(3,1) are

$$A_3 = -i\left(\cot\theta\cosh\beta\frac{\partial}{\partial\beta} - \sinh\beta\frac{\partial}{\partial\theta}\right) \tag{3.14}$$

and

$$L_\pm \equiv L_1 \pm iL_2 = e^{\pm i\phi}\left(\pm\left(\cosh\beta\frac{\partial}{\partial\theta} - \sinh\beta\cot\theta\frac{\partial}{\partial\beta}\right) + i\frac{\cot\theta}{\cosh\beta}\frac{\partial}{\partial\phi}\right). \tag{3.15}$$

Let us take, for a solution of (2.16) in the RMS, the form

$$\psi(x) = R(\rho)\Theta(\theta)B(\beta)\Phi(\phi). \tag{3.16}$$

From (3.7) and (3.9) it follows that

$$\frac{\partial^2}{\partial\phi^2}\Phi_m(\phi) = -\left(m + \frac{1}{2}\right)^2\Phi_m(\phi), \tag{3.17}$$

i.e.,

$$\Phi_m(\phi) = (1/\sqrt{2\pi})e^{i(m+(1/2))\phi}, \quad 0\leqslant\phi<2\pi, \tag{3.18}$$

where we have indexed the solutions by the separation constant m.

For the case m integer, $\Phi_m(2\pi+\epsilon) = -\Phi_m(\epsilon)$ (it is a two-valued function); we shall see that this is the interesting case. One must, in fact, use (3.18) for $m\geqslant 0$ and $\Phi_m(\phi)^*$ for $m<0$.

The operator Λ in (2.16) contains N^2; with (3.17), the action of N^2 on (3.16) is determined, for $m>0$, by

$$N^2 B_{mn}(\beta) = \left[\frac{\partial^2}{\partial\beta^2} + \tanh\beta\frac{\partial}{\partial\beta} + \frac{(m+\frac{1}{2})^2}{\cosh^2\beta}\right]B_{mn}(\beta) = (n^2 - \tfrac{1}{4})B_{mn}(\beta), \tag{3.19}$$

where n^2 labels the separation constant for the variable β. The term $(m+\frac{1}{2})^2$ is to be replaced by $(m-\frac{1}{2})^2 = (|m|+\frac{1}{2})^2$ for $m<0$. We study explicitly only the case $m\geqslant 0$ in what follows.

As a final step in our treatment of the Casimir operator Λ in (2.16), it follows from (3.7) and (3.19) that

$$\Lambda\Theta(\theta) = \left[-\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(n^2 - \frac{1}{4}\right)\right]\Theta(\theta). \tag{3.20}$$

IV. SOLUTIONS OF THE ANGULAR AND HYPERANGULAR EIGENVALUE EQUATIONS IN THE RMS

In this section, we shall solve the eigenvalue equations obtained by separation of variables. For the treatment of Eq. (3.19), it is convenient to introduce the variable

$$\zeta = \tanh\beta, \tag{4.1}$$

where we note that $-1\leqslant\zeta\leqslant 1$, and define

$$B_{mn}(\beta) = (1-\zeta^2)^{1/4}\hat{B}_{mn}(\zeta). \tag{4.2}$$

Equation (3.19) then becomes

$$(1-\zeta^2)\frac{\partial^2\widehat{B}_{mn}(\zeta)}{\partial\zeta^2}-2\zeta\frac{\partial\widehat{B}_{mn}(\zeta)}{\partial\zeta}+\left[m(m+1)-\frac{n^2}{1-\zeta^2}\right]\widehat{B}_{mn}(\zeta)=0. \tag{4.3}$$

The solutions of this well-known equation are the associated Legendre functions of the first and second kind,[24] $P_m{}^n(\zeta)$ and $Q_m{}^n(\zeta)$.

The normalization condition for the wave functions (3.16) [with the measure (3.10)] is

$$1=\int\rho^3\,d\rho\,d\phi\,d\beta\,d\theta\sin^2\theta\times\cosh\beta\,|R(\rho)|^2|\Theta(\theta)|^2|B(\beta)|^2|\Phi(\phi)|^2 \tag{4.4}$$

and hence we must require that

$$\int\cosh\beta\,|B(\beta)|^2\,d\beta<\infty. \tag{4.5}$$

In terms of the variable ζ, this condition is

$$\int_{-1}^{1}(1-\zeta^2)^{-1}|\widehat{B}(\zeta)|^2\,d\zeta<\infty. \tag{4.6}$$

For $\nu>0$, and $\mu=0,1,2,\ldots$, one has[25]

$$\int_{-1}^{1}(1-\zeta^2)^{-1}|P_{\mu+\nu}{}^{-\nu}(\zeta)|^2\,d\zeta=\frac{1}{\nu}\frac{\Gamma(1+\mu)}{\Gamma(1+\mu+2\nu)}. \tag{4.7}$$

We shall show in an Appendix that the solutions for $\mu=m+n$ integer build the irreducible representations for the O(2,1) subgroup, which constitute the admissible physical states. The associated Legendre functions of the second kind do not satisfy the normalization condition (4.6).

We may choose for the normalized solutions (it is sufficient to consider only $n\geqslant 0$)

$$\widehat{B}_{mn}(\zeta)=\sqrt{n}\,\sqrt{[\Gamma(1+m+n)/\Gamma(1+m-n)]}\times P_m{}^{-n}(\zeta), \tag{4.8}$$

where $m\geqslant n$.

The case $n=0$ must be treated with some care. For $n=0$, the associated Legendre functions $P_m{}^{-n}(\zeta)$ become the Legendre polynomials $P_m(\zeta)$. The end points of integration in (4.6), $\zeta=\pm 1$, correspond to $\beta\to\pm\infty$. In terms of integration on β, e.g., in (4.5), the factor $\cosh\beta=(1-\zeta^2)^{-1/2}$ in the measure is canceled by the square of the factor $(1-\zeta^2)^{1/4}$ in (4.2), so the integration appears as

$$\int_{-\infty}^{\infty}|\widehat{B}(\zeta)|^2\,d\beta. \tag{4.9}$$

The Legendre polynomials do not vanish at $\zeta=\pm 1$, and hence if $\widehat{B}$ and P_m are related by a finite coefficient, this integral would diverge. When n goes to zero, the wave function spreads along the hyperbola labeled by ρ, going asymptotically to the light plane; the probability density with respect to intervals of β becomes constant for large $|\beta|$. Events associated with the two particles may therefore be found (for sufficiently large separation in space) with 2 + 1 lightlike separation out to remote regions of the tangent planes. To maintain the normalization, the Legendre functions must be multiplied by a vanishing factor, and the probability goes pointwise to zero (the wave function approaches a generalized eigenfunction). We shall therefore use, for this case, the function defined by

$$\widehat{B}_m(\zeta)=\sqrt{\epsilon}(1-\zeta^2)^{\epsilon/2}P_m(\zeta), \tag{4.10}$$

where it is understood that the limit $\epsilon\to 0$ is to be taken after the computation of scalar products; the factor $(1-\zeta^2)^{\epsilon/2}$ is a residue of the formula (AS 8.6.6; see also 8.1.4)

$$P_m{}^{-n}(\zeta)=(-1)^n(1-\zeta^2)^{n/2}\frac{d^n}{d\zeta^n}P_m(\zeta). \tag{4.11}$$

From (3.12) and (3.13), we see that the operators of O(2,1) leave θ invariant. We show in Appendix B that the functions $\{\widehat{B}(\zeta)\Phi(\phi)\}$ constitute the discrete series of irreducible projective representation of O(2,1), and that it is not possible to use these representations to construct a ladder representation of O(3,1).

We now turn to the solution of Eq. (3.20). Let us define the variable

$$\xi=\cos\theta \tag{4.12}$$

and the function

$$\widehat{\Theta}(\theta)=(1-\xi^2)^{1/4}\Theta(\theta). \tag{4.13}$$

Equation (3.20) then becomes

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{d}{d\xi}\widehat{\Theta}(\xi)\right)+\left(l(l+1)-\frac{n^2}{1-\xi^2}\right)\widehat{\Theta}(\xi)=0, \tag{4.14}$$

where we have set

$$\Lambda=l(l+1)-\tfrac{3}{4}. \tag{4.15}$$

The solutions of Eq. (4.14) are proportional to the associated Legendre functions of the first and second kind, $P_l{}^n(\xi)$, $Q_l{}^n(\xi)$. For $n\neq 0$, the second kind of functions are not normalizable [the measure, according to (3.10) and (4.13) is the usual one for Legendre functions], and we therefore reject these. It follows from the requirement of unitarity for the representations of O(2,1) that we shall obtain, and normalizability, that l must be a non-negative integer (including 0) or positive half-integer.

To understand the geometrical and physical meaning of the quantum numbers l and n, consider the set of events parametrized by (3.1) with $\beta=0$ (these correspond to *equal time* correlations),

$$\begin{aligned}x^0&=0,\\x^1&=\rho\sin\theta\cos\phi,\\x^2&=\rho\sin\theta\sin\phi,\\x^3&=\rho\cos\theta.\end{aligned} \tag{4.16}$$

This set of events lies in a three-dimensional subspace parametrized by the usual spherical polar angles. The factor

$$Y_l{}^n(\theta,\phi)=(1/\sqrt{2\pi})e^{in\phi}\widehat{\Theta}_l{}^n(\cos\theta) \tag{4.17}$$

in the separated solution (3.16), where

$$\widehat{\Theta}_l{}^n(\theta)=\left(\frac{2l+1}{2}\frac{(l-n)!}{(l+n)!}\right)^{1/2}P_l{}^n(\cos\theta) \tag{4.18}$$

transforms under rotations according to

$$Y_l{}^n(\theta,\phi) = \sum_{n'} D^l{}_{nn'}(\eta_1,\eta_2,\eta_3) Y_l{}^{n'}(\theta',\phi'), \qquad (4.19)$$

where the $D^l{}_{nn'}$ are the Wigner rotation functions of Euler angles η_1,η_2,η_3.[26] Note that the Legendre functions of the second kind do not admit this interpretation. We recognize that the Casimir operator of the Lorentz group labels the irreducible representation of the rotation group here, and the Casimir operator of the O(2,1) subgroup labels the magnetic quantum number corresponding to orientations of the three-dimensional space parametrized in (4.16). A general point in the RMS is obtained from such a representative point by performing a boost in the (x^1,x^2) plane. For

$$x_\perp \equiv \sqrt{(x^1)^2 + (x^2)^2} = \rho \sin\theta, \qquad (4.20)$$

a boost with parameter β in the direction **x** results in

$$x'_\perp = x_\perp \cosh\beta,$$
$$x'^0 = x_\perp \sinh\beta, \qquad (4.21)$$

corresponding to the general form (3.1) (for some ϕ). Each event in the three-space parametrized by (4.16) can be mapped in this way into a corresponding set of points in the RMS. Conversely, each point in the RMS is projected into this three-space by taking $\beta = 0$.

A reorientation of the three-dimensional space of (4.16) by the transformation (4.19) admits the same construction. A mapping from points represented in the reoriented space into general points in the RMS can be carried out by a set of active boosts in the *new* (x^1,x^2) plane.

The result of the reorientation of the three-dimensional equal time space is a reorientation of the entire RMS. After the transformation, the new RMS is constructed, with boundary planes tangent to the light cone, oriented along the new z axis (we shall show in II that all possible orientations must be considered in the specification of the two-body state).

V. THE RADIAL EQUATION AND INVARIANT SPECTRUM

The remaining "radial" equation obtained from (2.16) after separation of the angular and hyperangular variables, taking into account (4.15), is

$$\left[\frac{1}{2m}\left(-\frac{\partial^2}{\partial\rho^2} - \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{l(l+1) - \frac{3}{4}}{\rho^2}\right) + V(\rho^2)\right]$$
$$\times R^{(a)}(\rho) = K_a R^{(a)}(\rho). \qquad (5.1)$$

Let us put

$$R^{(a)}(\rho) = (1/\sqrt{\rho})\hat{R}^{(a)}(\rho). \qquad (5.2)$$

Equation (5.1) then becomes

$$\frac{d^2\hat{R}^{(a)}(\rho)}{d\rho^2} + \frac{2}{\rho}\frac{d\hat{R}^{(a)}(\rho)}{d\rho} - \frac{l(l+1)}{\rho^2}\hat{R}^{(a)}(\rho)$$
$$+ 2m(K_a - V(\rho^2))\hat{R}^{(a)}(\rho) = 0, \qquad (5.3)$$

which is exactly of the form of the nonrelativistic spherically symmetric Schrödinger equation [the measure for the normalization of $\hat{R}$, according to (5.2) and (3.10), is just $\rho^2\, d\rho$]. The lowest mass eigenvalue for the case $V \propto 1/\rho$ occurs for the $l = 0$ state of the sequence $l = 0,1,2,3,\ldots$, and therefore the quantum number l plays a role analogous to that of orbital angular momentum in Eq. (5.3). In the interior region II, the spectrum of Λ is continuous.[27] In the full spacelike region, the last step of separation of variables associates the eigenvalues of Λ, which we have labeled with l, with a differential equation in the noncompact independent variable β [this can be seen from the structure of the parametrization (3.2) of the spacelike region, where β occurs in all four variables]. In this case,[28] $\Lambda = (l + \frac{1}{2})(l + \frac{3}{2}) - \frac{3}{4}$, for $l = 0,1,2,\ldots$, and hence the lowest achievable mass state is higher than the one we have obtained for wave functions with support in the RMS. This is the source of the spontaneous breaking of the O(3,1) space-time symmetry of the dynamical equations that selects the RMS subspace of the spacelike region.

For each nonrelativistic spherically symmetric potential problem, one obtains a corresponding direct action potential problem by the replacement of the relative radial coordinate r by ρ.

We shall argue below that the value of the full K operator (2.6) is usually determined (within a narrow interval) by intrinsic properties of the constituents. It then follows from the relation

$$K = P_a{}^2/2M + K_a \qquad (5.4)$$

that the mass spectrum of the two-body system is determined by the spectrum K_a of the reduced motion. The two-body invariant mass squared (center of mass energy squared) is then given by

$$s_a \equiv -P_a{}^2 = 2M(K_a - K); \qquad (5.5)$$

it is therefore quantized according to the spectrum of the relative motion, which coincides with the corresponding nonrelativistic energy spectrum.

Our argument that K is determined by intrinsic properties of the constituents is as follows. Transitions between bound state levels, involving changes in K_a, are induced by perturbation, such as coupling to electromagnetism. To treat such perturbations, we consider the addition of a τ-independent operator $\Delta V(x_1,x_2)$ that has non-negligible values in some limited space-time region (analogous to an adiabatic perturbation) near, for example, $x_i = 0$. Suppose, furthermore, that the wave function for the two-body system does not significantly overlap this perturbation for τ large and negative. It is in this range of τ values that we can consider the stationary bound state problem that we have studied here. At later τ, the wave function overlaps the perturbation, and transitions among the states of the stationary problem become possible. At large positive τ, the wave function no longer overlaps the ΔV and hence the system may again be found in a stationary state, perhaps different from the initial one (for example, radiation may have occurred). Since, however, ΔV is independent of τ, the value of K is conserved throughout the evolution. This situation is significantly different from the usual treatment of perturbations in nonrelativistic quantum theory, where the turning off and on of the perturbation in time causes transitions among values of the Hamiltonian operator. We therefore see that the relation between $P_a{}^2$ and K_a should be determined by (5.4), with a fixed value of K.

To determine this fixed value of K, we now suppose that the system is exposed to a τ-independent (but space-time-dependent) perturbation that brings the state of the system past the ionization point, if such a point exists. In this state, the constituent events may be separated by a large spacelike distance, where the potential is negligible (provided, as we shall see, a critical bound is not exceeded). Hence (see also Reuse[12])

$$K \simeq \frac{P^2}{2M} + \frac{p^2}{2m} = \frac{p_1^{\,2}}{2M_1} + \frac{p_2^{\,2}}{2M_2} \simeq -\frac{Mc^2}{2}, \tag{5.6}$$

where the last approximate equality follows from the assignment of each of the particles to a small interval in the neighborhood of its mass shell specified by its corresponding mass parameter M_i (if K varies over a small range, the two-body invariant mass squared varies over the same range; for each value of K, the quantization is determined by the discrete values of K_a). With (5.6), the mass squared spectrum (5.5) is

$$s_a \simeq M^2c^2 + 2MK_a. \tag{5.7}$$

If the nonrelativistic energy spectrum has values small compared to the particle rest masses, i.e., $|K_a| \ll Mc^2/2$, an *invariant* condition for nonrelativistic binding, the two-body center of mass energy spectrum is well approximated by

$$E_a \simeq Mc^2 + K_a - \tfrac{1}{2}K_a^2/Mc^2. \tag{5.8}$$

Up to the additive constant Mc^2, the center of mass energy thus coincides with the nonrelativistic energy spectrum to order $1/c^2$.

The families of functions Φ_m, B_{nm} for all values of m,n consistent with a given value of l form a degenerate set of solutions. The quantum numbers m,n of O(2,1) are a generalization of the magnetic quantum number that plays an analogous role in the corresponding nonrelativistic problem [the quantum number m changes under the action of the intrinsic O(2,1) subgroup].

It is interesting to note that the functions $\hat{R}$, $\hat{\Theta}$, and Φ have a correspondence interpretation. If the density $|\psi(x)|^2$ is used to study the expectation value of an observable that is a function on space-time that is independent of β [for example, a function of the O(2,1) invariant $x_1^{\,2} + x_2^{\,2} - x_0^{\,2}$], one may use the effective three-dimensional density given by [the probability of occurrence of an event in d^4x is $|\psi(x)|^2\, dx^0\, d^3x$]

$$\begin{aligned}\int |\psi(x)|^2 \frac{\partial x^0}{\partial \beta}\, d\beta &= \int |R(\rho)|^2 |\Theta(\theta)|^2 |\Phi(\phi)|^2 |B(\beta)|^2 \\ &\quad \times \rho \cosh\beta \sin\theta\, d\beta \\ &= (1/2\pi)|\hat{R}(\rho)|^2 |\hat{\Theta}(\theta)|^2,\end{aligned} \tag{5.9}$$

where $\hat{R}$, $\hat{\Theta}(\theta)$ coincide with the nonrelativistic wave functions (with the remaining measure $\rho^2\, d\rho \sin\theta\, d\theta\, d\phi$) for which ρ is the radial coordinate, and, as we found for the equal time correlation points at the end of Sec. IV, l is the orbital angular momentum, and n the magnetic quantum number [viz. (4.14) and (5.3)].

VI. SOME EXAMPLES

In this section, we give mass spectra for some exactly soluble problems, in particular, for the relativistic analog of the Coulomb potential, for which

$$V(\rho^2) = -Ze^2/\rho, \tag{6.1}$$

the four-dimensional space-time harmonic oscillator[29]

$$V(\rho^2) = \tfrac{1}{2}m\omega^2\rho^2, \tag{6.2}$$

and the relativistic analog of the three-dimensional square well potential, which has, in the relativistic case, a hyperboloidal boundary,[21] and for which

$$V(\rho^2) = \begin{cases} -U & \rho \leqslant a, \\ 0, & \rho > a. \end{cases} \tag{6.3}$$

In order to find the mass spectra and radial wave functions for these examples, it is not necessary to solve new differential equations. The radial equation (5.3) is exactly of the form of the corresponding nonrelativistic problem, and the solutions are known.

For the relativistic analog of the Coulomb potential, the relative mass spectrum is given by

$$K_a = -Z^2me^4/2\hbar^2(l+1+n_a)^2, \tag{6.4}$$

where $n_a = 0,1,2,\ldots$. The wave functions $\hat{R}\rho^{(a)}$ are the usual hydrogen functions[30]

$$\hat{R}_{n_a,l}(\rho) = C_{n_a,l}e^{-x/2}x^{l+1}L_{n_a}^{\,2l+1}(x), \tag{6.5}$$

where $L_{n_a}^{\,2l+1}$ are Laguerre polynomials. The variable x is defined by

$$x = (2Z\rho/a_0)/(n_a + l + 1), \tag{6.6}$$

where $a_0 = \hbar^2/me^2$, and

$$C^2_{n_a,l} = Z(n_a)!/(n_a + l + 1)^2(n_a + 2l + 1). \tag{6.7}$$

The *size* of the bound state, which is related to the atomic form factor, is measured according to the invariant ρ. For the lowest level, $n_a = l = 0$,

$$\langle\rho\rangle_{n_a = l = 0} = \tfrac{3}{2}a_0. \tag{6.8}$$

The total mass spectrum is then given by (5.7), i.e.,

$$s_{l,n_a} \simeq M^2c^2 - mMZ^2e^4/\hbar^2(n_a + l + 1)^2. \tag{6.9}$$

For the case that the nonrelativistic energy spectrum has value small compared to the particle rest masses, we may use the approximate relation (5.8) to obtain

$$\begin{aligned} E_a \simeq Mc^2 - Z^2\frac{me^4}{2\hbar^2(l+1+n_a)^2} \\ -\frac{1}{8}\frac{Z^4m^2e^8}{Mc^2\hbar^4(l+1+n_a)^4}.\end{aligned} \tag{6.10}$$

The lowest-order relativistic correction to the rest energy of the two-body system with Coulomb-like potential is therefore

$$\frac{\Delta(E_a - Mc^2)}{E_a - Mc^2} = \frac{Z^2\alpha^2}{4}\left(\frac{m}{M}\right)\frac{1}{(l+1+n_a)^2}. \tag{6.11}$$

For spinless atomic hydrogen ($Z = 1$), $\Delta(E - Mc^2) \simeq 9.7\times 10^{-8}$ eV, and $E - Mc^2 \simeq 13.6$ eV for the ground state. The relativistic correction is therefore of the order of one part in 10^8. It is, however, about 10% of the hyperfine splitting $\hbar c/21\ \text{cm} \simeq 9.4\times 10^{-7}$ eV. For positronium, $\Delta(E - Mc^2) \simeq 2\times 10^{-5}$ eV and $E - Mc^2 \simeq 6.8$ eV, so the relativistic correction is of the order of one part in 10^5. It is about 2% of the positronium hyperfine splitting $\frac{7}{6}\alpha^2$ Ry $\simeq 8.4\times 10^{-4}$ eV.[31]

For the four-dimensional harmonic oscillator, Eq. (5.3) has the form

$$\frac{d^2\hat{R}^{(a)}}{d\rho^2} + \frac{2}{\rho}\frac{d\hat{R}^{(a)}}{d\rho} + \left(\frac{2mK_a}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}\rho^2 - \frac{l(l+1)}{\rho^2}\right)\hat{R}^{(a)} = 0. \quad (6.12)$$

As for the nonrelativistic case, we make the transformation

$$\hat{R}^{(a)}(\rho) = x^{l/2}e^{-x/2}\, w^{(a)}(x), \quad (6.13)$$

where

$$x = (m\omega/\hbar)\rho^2, \quad (6.14)$$

to obtain

$$x\frac{d^2w^{(a)}}{dx^2} + \left(l + \frac{3}{2} - x\right)\frac{dw^{(a)}}{dx} + \frac{1}{2}\left(l + \frac{3}{2} - \frac{K_a}{\hbar\omega}\right)w^{(a)} = 0. \quad (6.15)$$

Normalizable solutions, the Laguerre polynomials $L_{n_a}^{\,l+1/2}(x)$, exist[30] when the coefficient of $w^{(a)}$ is a negative integer, i.e.,

$$K_a = \hbar\omega(l + 2n_a + \tfrac{3}{2}), \quad (6.16)$$

for $n_a = 0,1,2,\ldots$. The total mass spectrum is given by (5.7) (the choice of K is arbitrary here since there is no ionization point):

$$s_{l,n_a} = -2MK + 2M\hbar\omega(l + 2n_a + \tfrac{3}{2}). \quad (6.17)$$

For the case where the nonrelativistic energy spectrum has values small compared to K, which we surmise may be of the order of the particle rest masses,

$$E_a \simeq \sqrt{-2Mc^2K} + \hbar\omega\sqrt{(Mc^2/2|K|)}\,(l + 2n_a + \tfrac{3}{2}) - \tfrac{1}{2}(\hbar\omega)^2\sqrt{(Mc^2/8|K|^3)}\,(l + 2n_a + \tfrac{3}{2})^2. \quad (6.18)$$

Arbitrarily setting $K = -Mc^2/2$, one obtains

$$E_a \simeq Mc^2 + \hbar\omega\left(l + 2n_a + \frac{3}{2}\right) - \frac{1}{2}\frac{\hbar^2\omega^2(l + 2n_a + \frac{3}{2})^2}{Mc^2}. \quad (6.19)$$

Feynman, Kislinger, and Ravndal, Kim and Noz, and others[29] have studied the relativistic oscillator and obtained a positive spectrum [as in (6.17)] by imposing a subsidiary condition suppressing time excitations; although the mechanism is different, the restriction of the support of the wave functions to the O(2,1) invariant RMS plays an analogous role. No additional subsidiary condition is required; the set of solutions forms a complete orthogonal set in every Lorentz frame[32] (corresponding, in this case, to the induced representation to be described in II).

We now turn to the O(3,1) symmetric square well. In this case, the radial equation (5.3), with $V(\rho^2)$ given by (6.3), has solutions of the form (for $-U < K_a < 0$)[33]

$$\hat{R}^{(a)}(\rho) = \begin{cases} Aj_l(\sqrt{2m(K_a + U)/\hbar^2}\,\rho), & \rho \leqslant a, \\ Bh_l^{(1)}(i\sqrt{(-2mK_a)/\hbar^2}\,\rho), & \rho > a, \end{cases} \quad (6.20)$$

where j_l are spherical Bessel functions and $h_l^{(1)}$ are spherical Hankel functions of the first kind [the radial measure for $\hat{R}^{(a)}(\rho)$ is the same as for the nonrelativistic case]. Continuity of the wave function and its derivative with respect to ρ at the boundary $\rho = a$ provides the condition for the allowed values of K_a.

Let us call

$$\kappa_1 = \left(\frac{2m(K_a + U)}{\hbar^2}\right)^{1/2}, \quad \kappa_0 = \left(\frac{-2mK_a}{\hbar^2}\right)^{1/2}. \quad (6.21)$$

For $z_1 \equiv \kappa_1 a$, $z_0 \equiv \kappa_0 a \gg 1$, we may use the asymptotic forms

$$j^l(z) \sim (1/z)\cos(z - l\pi/2 - \pi/2), \qquad h_l^{(1)}(z) \sim (1/z)e^{i(z - l\pi/2 - \pi/2)}, \quad (6.22)$$

to obtain the eigenvalue conditions

$$-\cot\kappa_1 a \simeq \kappa_0/\kappa_1 \quad (l\ \text{even}), \qquad \tan\kappa_1 a \simeq \kappa_0/\kappa_1 \quad (l\ \text{odd}). \quad (6.23)$$

Since $\kappa_1^{\,2} + \kappa_0^{\,2} = 2mU/\hbar^2$, the large z_1, z_0 approximation requires that

$$\xi^2 \equiv (2mU/\hbar^2)a^2 \gg 1. \quad (6.24)$$

Defining

$$\epsilon = z_1 - \xi/\sqrt{2}, \quad (6.25)$$

the condition $\epsilon/\xi \ll 1$ then ensures, with (6.24), that z_0 and z_1 are both large. It then follows that

$$\frac{\kappa_0}{\kappa_1} = \left(\frac{\xi^2}{z_1^{\,2}} - 1\right)^{1/2} \simeq 1 - 2\sqrt{2}\,\frac{\epsilon}{\xi}. \quad (6.26)$$

For $\epsilon/\xi = 0$, solutions of (6.23) for l even are at $(4n - 1)/4\pi$ for integer $n \geqslant 1$, and for l odd, at $(4n + 1)/4\pi$ for integer $n \geqslant 0$. Expanding the trigonometric functions in the neighborhood of these values, and comparing with (6.25), we obtain

$$z_1(n) \simeq n\pi \mp \pi/4 - \sqrt{2}\epsilon/\xi$$

for l even or odd. Since, however, ϵ depends on z_1, we may substitute (6.25) and solve for $z_1(n)$, obtaining

$$z_1(n) \simeq (1 - \sqrt{2}/\xi)(n\pi + 1 \mp \pi/4) \simeq n\pi, \quad (6.27)$$

where $n\pi \gg 1$. Since $\epsilon/\xi = z_1/\xi - 1/\sqrt{2} \ll 1$, our solution is valid for values of n such that $n\pi/\xi \simeq 1/\sqrt{2}$.

For this set of high levels, the spectrum is given by

$$K_a \simeq -\{U - n_a^{\,2}\pi^2\hbar^2/2ma^2\}. \quad (6.28)$$

From (5.8), it follows that

$$E_a \simeq Mc^2 - \left(U - \frac{n_a^{\,2}\pi^2\hbar^2}{2ma^2}\right) - \frac{1}{2Mc^2}\left(U - \frac{n_a^{\,2}\pi^2\hbar^2}{2ma^2}\right)^2 \quad (6.29)$$

and the lowest-order relativistic correction to the relativistic spectrum is

$$\frac{\Delta(E_a - Mc^2)}{E_a - Mc^2} \simeq \frac{1}{2Mc^2}\left(U - \frac{n_a^{\,2}\pi^2\hbar^2}{2ma^2}\right). \quad (6.30)$$

The result (6.28) illustrates in a simple and explicit way a rather remarkable relativistic effect. Since an indefinite increase in the well depth U is in the framework of the approximation we have made in arriving at (6.28), which can be written alternatively as

$$K_a \simeq -\frac{\hbar^2}{2ma^2}\frac{\xi^2}{2}\left(1 - 2\sqrt{2}\,\frac{\epsilon(n)}{\xi}\right),$$

it is evident that the center of mass energy squared,

$$s_a = 2M(K_a - K), \tag{6.31}$$

can eventually become negative for any fixed value of K, for example, $-Mc^2/2$, as asserted in (5.6) (in this case, for $U \gtrsim Mc^2$). The argument leading to $K \simeq -Mc^2/2$ cannot, therefore, be justified in case the well depth U exceeds Mc^2 by a significant amount. This argument assumed that at, or above, the ionization point, the two particles may separate, and that the corresponding free motion can be consistent with $p_1{}^2 \sim -M_1{}^2c^2$ and $p_2{}^2 \sim -M_2{}^2c^2$. This would imply that the interpretation of the bound state as a composite system of the two particles with normal asymptotic behavior could be tenable. In this example, however, we see that if the potential well is sufficiently deep, this argument must fail, and ionization results in quasifree particle states for which the asymptotic values of $p_1{}^2$, $p_2{}^2$ must depend on the well depth (since the potential is bounded by a hyperboloid in space-time, only asymptotically approaching the light cone, it may be argued that unless there is compact support in t, there is always some small overlap of the wave function with the potential well no matter how large the spacelike separation). The drift of the particles out of the interaction region may be entirely suppressed, in fact, if there is a mechanism (such as self-energy) that induces a strong spectral enhancement of the asymptotic states of the two particles in the neighborhood of a definite value of the mass. In any case, the notion of a bound state as a composite of two particles with intrinsic properties determined in their free states becomes untenable when the binding potential is sufficiently strong. In this case, K must be treated as an unknown parameter, to be fixed to the observed spectrum. In the nonrelativistic limit, for which $c \to \infty$ (relative to all velocities), there is no U sufficiently large for this phenomenon to occur, and hence it must be understood as a relativistic effect.

The same phenomenon occurs for the Coulomb type potential, e.g., for Z sufficiently large, as can be seen from (6.9). The assignment of $K \simeq -Mc^2/2$ becomes untenable at

$$Z \gtrsim (M/\sqrt{M_1M_2})(1/\alpha). \tag{6.32}$$

If $M_1 \ll M_2$, the condition (6.32) becomes

$$Z \gtrsim \sqrt{(M_2/M_1)}\,(1/\alpha). \tag{6.33}$$

so that for one electron in the Coulomb field of a nucleus (for $M_2 \sim 2ZM_p$) the bound on Z for tenability of compositeness is very high ($\sim 5\times 10^5$).

For a system of two particles of equal mass parameter,

$$Z \gtrsim 2/\alpha, \tag{6.34}$$

which is of the order of magnitude of the value at which the spectrum of the Dirac equation becomes unstable. For a Coulomb-type strong interaction, where $\alpha \sim 1$, one sees that a simple picture of compositeness becomes questionable for any $Z \geqslant 1$.

VII. SUMMARY AND DISCUSSION

The eigenvalue equation for reduced motion (2.9), where $V(\rho^2)$ is an O(3,1) symmetric potential function, can be solved by separation of variables in the angular and hyperbolic angular coordinates (3.1) that range over the restricted Minkowski space (RMS) shown in Figs. 1 and 2 (the relativistic Coulomb-like problem can also be separated in hyperparaboloidal coordinates in this region; we shall discuss this procedure, along with the dynamical group of relativistic hydrogen, making use of a relativistic Runge–Lenz vector, elsewhere). The sequence of separation equations is in order ϕ,β,θ,ρ where β is a hyperbolic variable [in the full spacelike region, described by (3.2), the order of separation is ϕ,θ,β,ρ]. After the last stage of separation of variables, we are left with an equation in ρ that determines the spectrum. In the case of the full spacelike region, this radial equation depends on the separation constant for the β dependence; in the RMS, it is the separation constant for the θ dependence [which corresponds to the O(3,1) Casimir operator] that enters. In the nonrelativistic limit, O(3,1) is deformed to O(3) (the relative variables t, p^0 vanish in this limit), and the eigenvalues of the O(3,1) Casimir operator become eigenvalues of the O(3) Casimir operator, i.e., the angular momentum. Separation of variables in the RMS therefore has a clear correspondence to the nonrelativistic problem. The spectrum one finds in the full spacelike region and in the RMS are different. The lowest bound state in the RMS is lower than that found in the full spacelike region for $V \propto 1/\rho$, the relativistic generalization of the Coulomb potential.

Cook[15] has studied an equation similar to (2.9) with gauge invariant form for the electromagnetic interaction. In his approximations, the problem can be put into correspondence with the relativistic Coulomb potential problem we have studied. He obtains a mass spectrum proportional to a quantity of the form $-(n_a + l + \frac{1}{2})^{-2}$. This denominator is always half-integer squared and does not go to the Balmer form in the nonrelativistic limit. Its lowest value is higher than that of (6.4). As pointed out by Cook, the replacement of one of his quantum numbers (l) by a half-integer to compensate for this problem would lead to incorrect angular dependence.

Cook furthermore estimated the relativistic corrections both for Bohr–Sommerfeld quantization of his classical solutions (in the full spacelike region) and for a modified version of the treatment of the differential equations in the quantum case with extended sources admitting half-integer values for his analog of our n_a. He found that the $(\alpha/n)^4$ term [which we obtained in (6.10)] cannot be accounted for in his treatment.

The angular functions $P_l{}^n(\cos\theta)$ appearing in the solutions of the O(3,1) symmetric problem are in precise correspondence with those of the nonrelativistic case. The quantum number l specifies the O(3,1) Casimir operator, but it occurs in the relativistic radial equation in the same way that orbital angular momentum enters the nonrelativistic radial equation; in the nonrelativistic limit, O(3,1) is deformed to O(3), and l becomes the orbital angular momentum. The quantum number n specifies the O(2,1) Casimir operator; it becomes the magnetic quantum number in the nonrelativistic limit. The mass levels for the relativistic problem are degenerate in the O(2,1) quantum number, but not, in general, in l.

The restriction of the relative coordinates to the RMS corresponds to a restricted range of correlations available to the two events propagating in a bound state, i.e., to the range of $x_1^\mu - x_2^\mu$ available at each τ. We have assumed, in computing the full spectrum with functions whose support is restricted to the RMS, that this correlation is maintained for excited states as well.

The selection of wave functions defined on the O(2,1) invariant RMS corresponds to spontaneous symmetry breaking of the O(3,1) Lorentz invariance of the dynamical differential equation. The representations of O(3,1) generated by the solutions of the differential equation are, as we shall show in II, of induced type. Under the action of the full O(3,1), the solutions defined on the RMS specified by a spacelike unit vector (e.g., a unit vector along the z axis, as for the coordinate system used in this paper) undergo a Wigner transformation under the little group O(2,1), and are transported along an orbit parametrized by this spacelike vector whose range, under Lorentz transformation, is a single sheeted hyperboloid.

Due to the success of our choice of the RMS for the relativistic Coulomb problem, we have assumed that this region provides the correct correlations for two-body bound state O(3,1) symmetric potential problems in general, and a few examples are worked out.

Previous treatments of the relativistic harmonic oscillator problem[29] have imposed a subsidiary condition to ensure that timelike excitations are suppressed. Imbedding the bound state in the RMS instead of the full spacelike region eliminates the need for this condition. It replaces an explicit constraint by the introduction of coordinates whose free variation has sufficient structure to ensure that all excitations lie within a Hilbert space that has a consistent physical interpretation (positive norm); the spectrum corresponds to the excitations of just three harmonic degrees of freedom.

The relative mass eigenvalues of the relativistic square well potential problem were computed for a range of high levels for which the transcendental equations for the spectrum can be solved explicitly. It was found that, with the condition that the total K of the system takes on its asymptotic expected value for free particles approximately on mass shell above the ionization point, the well depth can be chosen sufficiently deep (in this case, $U \gtrsim Mc^2$) that the total invariant rest energy squared of the system can become negative. The assumption that the constituent particles behave asymptotically, above ionization, as free, therefore becomes untenable. A similar phenomenon occurs for Coulomb-type binding [at $Z \gtrsim (1/\alpha)M/\sqrt{M_1M_2}$]. For particles of equal mass, this criterion is met at the order of magnitude at which the Dirac spectrum becomes unstable, but for an electron in the Coulomb-type field of a heavy nucleus, the bound is very high ($\sim 5\times 10^5$). For strong coupling, of the order $\alpha \sim 1$, the assumption that the constituents can be assigned on-shell values asymptotically becomes questionable for any $Z \geqslant 1$.

We emphasize that this critical value of the binding does not correspond to an instability in the spectrum of the dynamical evolution operator. It implies a limit to the depth of binding for which the simple notion of a bound state as a composite system of two particles with intrinsic properties determined as independent free particles above ionization becomes untenable. In the nonrelativistic limit, no bounded potential can produce this phenomenon, and hence it must be understood as a relativistic effect.

The solution of the problem of the relativistic bound state in an O(3,1) symmetric potential that we have given provides a *mass* spectrum that is the same as the corresponding nonrelativistic Schrödinger energy spectrum; this mass spectrum, up to the additive constant Mc^2, becomes the energy spectrum, and the wave functions acquire their usual nonrelativistic interpretation (for which l becomes the angular momentum, and n the magnetic quantum number), in the nonrelativistic limit. The structure of the theory therefore satisfies a correspondence principle.

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APPENDIX: DISCRETE SERIES OF IRREDUCIBLE REPRESENTATIONS OF O(2,1), THE QUANTUM NUMBERS, AND THE NONEXISTENCE OF A LADDER REPRESENTATION FROM THIS SERIES FOR O(3,1)

The representations of SO(2,1) and its double covering SU(1,1) have been studied by many authors.[22,34] Bargmann,[22] in particular, has discussed the basis functions with support in sector II, where $x_0^2 - x_\perp^2 \geqslant 0$. We are interested in the wave functions on a Hilbert space in the RMS, where $x_0^2 - x_\perp^2 \leqslant 0$.

We show explicitly in this Appendix that the solutions (3.18) and (4.8) that we have obtained for the β,ϕ parts of the differential equation (2.16) constitute the double-valued discrete series of irreducible projective representations of O(2,1).

The operators H_- and H_+ defined in Eq. (3.12) act as raising and lowering operators for the index m, since

$$[L_3,\ H_\pm] = \pm H_\pm . \tag{A1}$$

We now show that the $k=0$ element of the sequence (it is convenient to replace m by $n+k$)

$$\chi_{n+k}{}^{-n}(\zeta,\phi) \equiv B_{n+k,n}(\beta)\Phi_{n+k}(\phi) = (1-\zeta^2)^{1/4}\hat{B}_{n+k,n}(\zeta)\Phi_{n+k}(\phi) \tag{A2}$$

satisfies

$$H_-\chi_n{}^{-n}(\zeta,\phi) = 0. \tag{A3}$$

In terms of the variable $\zeta = \tanh\beta$,

$$H_\pm = e^{\pm i\phi}\left(-i(1-\zeta^2)\frac{\partial}{\partial\zeta} \pm \zeta\frac{\partial}{\partial\phi}\right), \tag{A4}$$

and Eq. (A3) becomes

$$\left[(1-\zeta^2)\frac{\partial}{\partial\zeta} + \left(n+\frac{1}{2}\right)\zeta\right]\chi_n{}^{-n}(\zeta,\phi) = 0. \tag{A5}$$

Using the relation (AS 8.6.17),

$$P_n{}^{-n}(\zeta) = \frac{1}{\Gamma(1+n)}\frac{(1-\zeta^2)^{n/2}}{2^n},$$

and (4.8), (A3) follows immediately.

We now study the action of H_+ on this lowest state:

$$\begin{aligned} H_+\chi_n{}^{-n}(\zeta,\phi) &= e^{i\phi}\left(-i(1-\zeta^2)\frac{\partial}{\partial\zeta} + i\zeta\left(n+\frac{1}{2}\right)\right)\chi_n{}^{-n}(\zeta,\phi) \\ &= i\sqrt{2n+1}\ \chi_{n+1}{}^{-n}(\zeta,\phi). \end{aligned} \tag{A6}$$

In general,

$$\begin{aligned} H_+\chi_{n+k}{}^{-n}(\zeta,\phi) &= i\sqrt{n}\left(\frac{\Gamma(1+k+2n)}{\Gamma(1+k)}\right)^{1/2}(1-\zeta^2)^{1/4} \\ &\quad\times\left\{(n+k+1)\zeta P_{n+k}{}^{-n}(\zeta) - (1-\zeta^2)\frac{\partial}{\partial\zeta}P_{n+k}{}^{-n}(\zeta)\right\}\Phi_{n+k+1}(\phi). \end{aligned} \tag{A7}$$

It follows from (AS 8.5.3) and (AS 8.5.4) that

$$\begin{aligned} (1-\zeta^2)\frac{\partial}{\partial\zeta}P_{n+k}{}^{-n}(\zeta) &= (n+k+1)\zeta P_{n+k}{}^{-n}(\zeta) \\ &\quad - (2n+k+1)P_{n+k+1}{}^{-n}(\zeta), \end{aligned}$$

and hence (A7) becomes

$$H_+\chi_{n+k}{}^{-n}(\zeta,\phi) = i\sqrt{(k+1)(2n+k+1)}\,\chi_{n+k+1}{}^{-n}(\zeta,\phi). \tag{A8}$$

The Hermiticity of A_1, A_2 then implies that

$$\begin{aligned} (\chi_{n+k}{}^{-n},\ H_-\chi_{n+k+1}{}^{-n}) &= (\chi_{n+k+1}{}^{-n}, H_+\chi_{n+k}{}^{-n})^* \\ &= -i\sqrt{(k+1)(2n+k+1)} \end{aligned}$$

and hence [since H_- can only lower the k value, according to (A1)]

$$H_-\chi_{n+k+1}{}^{-n} = -i\sqrt{(k+1)(2n+k+1)}\,\chi_{n+k}{}^{-n}. \tag{A9}$$

This result is, of course, consistent with the commutation relation

$$[H_+, H_-] = -2L_3, \tag{A10}$$

which follows from the formal commutation relations of the Lorentz group algebra

$$[M^{\mu\nu}, M^{\alpha\beta}] = -i(g^{\nu\alpha}M^{\mu\beta} - g^{\beta\mu}M^{\alpha\nu} - g^{\nu\beta}M^{\mu\alpha} + g^{\alpha\mu}M^{\beta\nu}). \tag{A11}$$

For the O(2,1) subalgebra it follows from (A8) and (A9) that

$$(H_+H_- - H_-H_+)\chi_{n+k}{}^{-n} = -2(n+k+\tfrac{1}{2})\chi_{n+k}{}^{-n}. \tag{A12}$$

We now note that the complex conjugate of $\{\chi_{n\pm k}{}^{-n}\}$ transforms under $H_\pm$ in a similar way. We obtain in this way another, inequivalent, representation with the same value of the Casimir operator for O(2,1) [these elements correspond to the replacement of $m+\frac{1}{2}$ by $m-\frac{1}{2}$ for $m<0$ in (3.18) and (3.19); as we remarked after (3.19), we shall continue to consider m as positive]. Since the functions $B_{n+k,n}$ are real, we consider

$$\chi_{n+k}{}^{-n*}(\zeta,\phi) = (1-\zeta^2)^{1/4}\hat{B}_{n+k,n}(\zeta)\Phi_{n+k}{}^*(\phi). \tag{A13}$$

Since, according to (A4),

$$H_+{}^* = -H_-, \tag{A14}$$

it follows from (A2) that

$$H_+\chi_n{}^{-n*}(\zeta,\phi) = 0, \tag{A15}$$

and hence there is a sequence with a highest element. The Clebsch–Gordan coefficients are determined by (A8) and (A9). Using (A14), one obtains

$$\begin{aligned} H_-\chi_{n+m}{}^{-n*}(\zeta,\phi) &= i\sqrt{(k+1)(2n+k+1)}\,\chi_{n+k+1}{}^{-n*}(\zeta,\phi), \\ H_+\chi_{n+k+1}{}^{-n*}(\zeta,\phi) &= -i\sqrt{(k+1)(2n+k+1)}\,\chi_{n+k}{}^{-n*}(\zeta,\phi). \end{aligned} \tag{A16}$$

In fact, this complementary representation corresponds to charge conjugation. Since the operators A_1, A_2, L_3 are Hermitian, complex conjugation is equivalent to the transpose. Replacing the operators by their negative transpose, which corresponds to group theoretical charge conjugation (to be denoted by C), leaves the commutation relations invariant. Under this action,

$$\begin{aligned} &H_-{}^C = -H_+{}^* = H_-, \quad H_+{}^C = -H_-{}^* = H_+, \\ &L_3{}^C = -L_3{}^* = L_3, \end{aligned} \tag{A17}$$

where the last follows from (3.13). The two representations are therefore related by charge conjugation.

The O(2,1) Casimir operator defined in (3.12) is, in this set of representations, given by

$$\begin{aligned} N^2 = L_3{}^2 - A_1{}^2 - A_2{}^2 &= L_3{}^2 - \tfrac{1}{2}(H_+H_- + H_-H_+) \\ &= L_3(L_3 - 1) - H_+H_-. \end{aligned} \tag{A18}$$

With the help of (A8) and (A9) [or, correspondingly, (A16)], and the action of L_3, one obtains, as required by (3.19),

$$N^2 = n^2 - \tfrac{1}{4}. \tag{A19}$$

The unitary irreducible representations of O(2,1) are single or double valued, and hence m must be half-integer or integer, the latter corresponding to the double-valued representation. As we have seen, k is integer valued, and therefore n must be half-integer or integer, also. Normalizability conditions on the associated Legendre functions then require that l be, respectively, half-integer or integer. As we have remarked in Sec. V, the lowest mass state (for the soluble problems we have considered) corresponds to $l=0$, and hence we shall only consider the integer values of l. This is consistent with our identification of the spectrum of K_a, of (5.3), with that of the corresponding nonrelativistic potential problem, and the correct behavior of the angular functions in that limit. We are therefore dealing with the double-valued representations of O(2,1).

In the following, we show that the operators A_3 and $L_\pm$ [which are not in the algebra of O(2,1)] move the set of eigenfunctions we have found out of the Hilbert space.

In terms of the variables ξ,ζ,ϕ, it follows from (3.14) and (3.15) that

$$A_3 = -i\left(\xi\left(\frac{1-\zeta^2}{1-\xi^2}\right)^{1/2}\frac{\partial}{\partial\zeta} + \zeta\left(\frac{1-\xi^2}{1-\zeta^2}\right)^{1/2}\frac{\partial}{\partial\xi}\right) \tag{A20}$$

and

$$L_\pm = -e^{\pm i\phi}\left(\pm\left(\frac{1-\xi^2}{1-\zeta^2}\right)^{1/2}\frac{\partial}{\partial\xi} \pm \left(\frac{1-\zeta^2}{1-\xi^2}\right)^{1/2}\zeta\xi \times\frac{\partial}{\partial\zeta} - i\left(\frac{1-\zeta^2}{1-\xi^2}\right)^{1/2}\xi\frac{\partial}{\partial\phi}\right). \tag{A21}$$

The action of these operators on the normalized eigenstates discussed above does not lead to a ladder representation for O(3,1) [unlike the case of the reduction O(3,1)⊃O(3) (Refs. 35 and 36)]. Let us study, for example, the action of A_3 on the normalized wave function $f_{l,m}{}^{-n}$ (taking again $m=n+k$),

$$f_{l,n+k}{}^{-n}(\theta,\beta,\phi) = \Theta_l{}^n(\theta)B_{n+k,n}(\beta)\Phi_{n+k}(\phi), \tag{A22}$$

where

$$\Theta_l{}^n(\theta) = \left[\left(\frac{2l+1}{2}\right)\left(\frac{(l-n)!}{(l+n)}\right)\right]^{1/2} P_l{}^n(\xi)(1-\xi^2)^{-1/4}. \tag{A23}$$

With the definitions (4.2) and (4.8), and (A20), we obtain

$$\begin{aligned} A_3 f_{l,n+k}{}^{-n}(\theta,\beta,\phi) &= -i\sqrt{n}\left(\frac{\Gamma(1+2n+k)}{\Gamma(1+k)}\right)^{1/2}\left[\left(\frac{2l+1}{2}\right)\left(\frac{(l-n)!}{(l+n)!}\right)\right]^{1/2}\Phi_{n+k}(\phi)\left\{\xi\left(\frac{1-\zeta^2}{1-\xi^2}\right)^{3/4}P_l{}^n(\xi)\frac{\partial}{\partial\zeta}P_{n+k}{}^{-n}(\zeta)\right. \\ &\quad \left. +\zeta\left(\frac{1-\xi^2}{1-\zeta^2}\right)^{1/4}P_{n+k}{}^{-n}(\zeta)\frac{\partial}{\partial\xi}P_l{}^n(\xi)\right\}. \end{aligned} \tag{A24}$$

Using identities for the associated Legendre functions,[36,37] we may write (A24) as

$$\begin{aligned} A_3 f_{l,n+k}{}^{-n}(\theta,\beta,\phi) &= \frac{i}{2}\left\{\left(\frac{k(2n+k+1)(l-n)(l+n+1)}{n(n+1)}\right)^{1/2} \times f_{l,n+k}{}^{-n-1}(\theta,\beta,\phi)\right. \\ &\quad \left. -\left(\frac{(1+k)(2n+k)(l+n)(l-n+1)}{n(n-1)}\right)^{1/2} \times f_{l,n+k}{}^{-n+1}(\theta,\beta,\phi,)\right\}. \end{aligned} \tag{A25}$$

This recursion relation is similar in form to that obtained from the ladder representation based on O(3) (Ref. 36) which, for the spinless case ($l_1'=0$, so that $A_{l'}=0$), is given by[36]

$$A_3\xi_{l',m'} = C_{l'}\sqrt{l'^2-m'^2}\,\xi_{l'-1,m'} - C_{l'+1} \times\sqrt{(l'+1)^2-m'^2}\,\xi_{l'+1,m'}, \tag{A26}$$

where

$$C_{l'} = i\left(\frac{(l'^2-l_0'^2)}{4l'^2-1}\right)^{1/2}, \quad A_{l'}=0,$$

and $m' = -l', -l'+1,\ldots,l'$, $l'=l_0', l_0'+1,\ldots$.

The correspondence can be easily seen by recalling that $k=m-n$, and that $n+\frac{1}{2}$ [which determines the value of the O(2,1) Casimir operator] should be put into correspondence with the angular momentum quantum number l' of O(3). Hence, in the sense of this correspondence,

$$k(2n+k+1) \sim m'^2 - l'^2, \tag{A27}$$

where $m'\sim m+\frac{1}{2}$. In the second coefficient, $k\rightarrow k-1$ is equivalent to $l'\rightarrow l'+1$. The second pair of factors in the radical of the first coefficient of (A25) corresponds to

$$(l-n)(l+n+1) \sim l_0'^2 - l'^2, \tag{A28}$$

where l_0', the lowest angular momentum of the corresponding tower of O(3) representations, is identified with $l+\frac{1}{2}$ [we are considering the $(l+\frac{1}{2},0)$ double-valued representation]. The corresponding factors of the second term are similarly obtained by the substitution $l'\rightarrow l'+1$, inducing $n\rightarrow n-1$.

The recursion relation (A25), however, cannot be used to generate a proper ladder representation based on O(2,1), since, for example, applying A_3 to $f_{l,n+k}{}^{-n}$ for $n=1$ produces a term proportional to $f_{l,1+k}{}^0$. As we have pointed out in the discussion following Eq. (4.8), we can consider this function to be normalizable by the procedure of using the function $f_{l,1+k}{}^{-\epsilon}$ and taking the limit $\epsilon\rightarrow 0$ after integration.

The compensation for the singularity generated by the measure (3.10) for this function is obtained from the normalization factor in (4.8). The explicit appearance of the singularity $1/\sqrt{n-1}$ in (A25) for $n \to 1$ is precisely from this normalization. Since no such regularization procedure (by normalization) is available after operation with A_3, we see that this operator is not defined on $f_{l,1+k}{}^{-1}$; it shifts this function out of the Hilbert space.

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