A FAMILY OF PERIODIC SOLUTIONS OF THE PLANAR THREE-BODY PROBLEM, AND THEIR STABILITY

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Abstract. We describe a one-parameter family of periodic orbits in the planar problem of three bodies with equal masses. This family begins with Schubart's (1956) rectilinear orbit and ends in retrograde revolution, i.e. a hierarchy of two binaries rotating in opposite directions. The first-order stability of the orbits in the plane is also computed. Orbits of the retrograde revolution type are stable; more unexpectedly, orbits of the 'interplay' type at the other end of the family are also stable. This indicates the possible existence of triple stars with a motion entirely different from the usual hierarchical arrangement.

1. Introduction

In a previous paper (Hénon, 1974, herafter called I), it was shown that periodic solutions of the planar three-body problem with given masses form one-parameter families in general. Segments of two families were found numerically, starting from two particular orbits computed by Standish (1970) and Szebehely (1970).

The next step was to try to obtain an entire family of periodic orbits, rather than just a segment. We could have tried to extend the families of Paper I to their natural ends; this, however, would have been time-consuming because the orbits are rather complicated; besides, these families were arbitrarily selected examples, without anything very remarkable about them, among a probably large number of similar families having roughly the same degree of complexity. Therefore we preferred to start afresh and to explore a simpler and more significant family. We took as starting point a periodic orbit computed by Schubart (1956). The corresponding family was found to consist of comparatively simple orbits, and could be followed without trouble to its natural end – which came as something of a surprise. As will be seen, this family has some remarkable properties. A detailed description of the family is given in Section 2. After the present work had been completed, we were informed that Broucke (1975) had independently computed a part of the same family.

Many applications, either theoretical or practical, require a knowledge of the stability properties of periodic orbits. We describe in Section 3 a general method for the computation of the first-order stability of plane periodic solutions of the three-body problem. A similar method was independently devised by Hadjidemetriou (1975). Section 4 presents the stability properties of our family of periodic orbits. Section 5 is devoted to comments and applications.

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2. A New Family of Periodic Orbits

The starting point of our family is the remarkable periodic orbit computed by Schubart (1956; Table 3 and Figure 5), for the case of three equal masses:

$$m_1 = m_2 = m_3 = \tfrac{1}{3}. \tag{1}$$

As in I, we normalize the total energy to the value

$$E = -\tfrac{1}{2}(m_1 m_2 + m_2 m_3 + m_3 m_1) \tag{2}$$

or $E = -\frac{1}{6}$ in the present case. Schubart's orbit is rectilinear: the three bodies move on the x-axis. At time $t=0$, bodies 1 and 2 have a collision at $x_1 = x_2 = -a$, with $a = 0.717\,560\,83\ldots$, while body 3 is motionless at $x_3 = 2a$. Then x_1 and x_3 decrease, x_2 increases, until at $t = T/2$ bodies 2 and 3 have a collision at $x_2 = x_3 = a$, while body 1 is at $x_1 = -2a$ with zero velocity. After $t = T/2$ the motion is reversed and at $t = T = 4.698\,083\,80\ldots$ the system is back to its initial state.

According to the general predictions of Paper I, Schubart's periodic orbit should be isolated inside the set of rectilinear orbits; but if we consider the larger set of *planar* orbits, then Schubart's orbit should be a member of a continuous one-parameter family of periodic orbits. These expectations were fully confirmed by the numerical results, and the family was easily found, following the general method described in I. We call it *family m*, for reasons which will become apparent later. Its members have a non-zero angular momentum A in general, and also a non-zero rotation angle Φ after one period. (We use here Φ instead of φ in Paper I, so that all global parameters of the orbit are represented by a capital letter.) The collisions disappear as soon as Schubart's orbit is left. We assume, as in I, that the x-axis is parallel to the direction from 2 to 3 at time $t=0$ and we represent by x_i, y_i, u_i, v_i the coordinates of position and velocity of body i. All orbits of the family are then found to be symmetrical with respect to the x-axis:

$$\begin{aligned} &x_i(-t) = x_i(t), \qquad y_i(-t) = -y_i(t), \\ &u_i(-t) = -u_i(t), \qquad v_i(-t) = v_i(t), \qquad i = 1, 2, 3. \end{aligned} \tag{3}$$

In particular:

$$y_i(0) = 0, \qquad u_i(0) = 0, \quad i = 1, 2, 3. \tag{4}$$

So at time $t=0$ the three bodies are on the x-axis and move perpendicularly to it. Let us consider, as in I, a system of axes (X, Y) which rotates with a constant angular velocity $\omega = \Phi/T$ and coincides with (x, y) for $t=0$; there is also

$$\begin{aligned} &X_i(-t) = X_i(t), \qquad Y_i(-t) = -Y_i(t), \\ &\dot{X}_i(-t) = -\dot{X}_i(t), \qquad \dot{Y}_i(-t) = \dot{Y}_i(t), \quad i = 1, 2, 3, \end{aligned} \tag{5}$$

and

$$Y_i(0) = 0, \qquad \dot{X}_i(0) = 0, \quad i = 1, 2, 3. \tag{6}$$

Moreover, the orbits are closed in the rotating system:

$$X_i(t + T) = X_i(t), \quad \text{etc.} \tag{7}$$

It follows immediately from (5) and (7) that

$$Y_i(T/2) = 0, \qquad \dot{X}_i(T/2) = 0, \quad i = 1, 2, 3. \tag{8}$$

This is the well-known symmetry exhibited by most known periodic orbits in the restricted problem (see Szebehely, 1967), and also by the periodic orbits of the general three-body problem computed by Broucke and Boggs (1975), Broucke (1975), Hadjidemetriou (1975), Hadjidemetriou and Christides (1975), Bozis and Christides (1975). It would therefore be possible to compute only one half of the orbits (cf. Hadjidemetriou, 1975); however, since our program is designed for the general case of periodic orbits without any symmetry, we have not used this simplification, and the orbits were effectively integrated over one whole period. The conditions (4) were not enforced on the initial coordinates; rather, the program adjusted all four initial coordinates x_1, y_1, u_1, v_1 so as to obtain a periodic orbit, as explained in I. In other words, the computer was unaware of the symmetry of the orbits. Thus, the extent to which the computed values of $y_i(0)$ and $u_i(0)$ deviated from zero could be used as another accuracy test. This deviation was found to be of the order of 10^{-14}.

Table I lists quantities of interest for a number of orbits of the family. Initial values are given for x_1, v_1, x_2, v_2 only; the remaining initial coordinates are given by (4) and

$$x_3 = -(x_1 + x_2), \qquad v_3 = -(v_1 + v_2). \tag{9}$$

Also listed are the period T and the rotation angle Φ. The quantities k_1 and k_2 will be explained in Section 3.

Figure 1 represents a few selected orbits, identified by their number in Table I, and represented in the fixed system of reference (x, y). Filled symbols represent the initial positions; open symbols represent the final positions. One full period is shown, from $t=0$ to $t=T$; therefore the final positions are derived from the initial positions by a rotation of angle Φ around the origin (marked by a cross). More generally, the motion between $t=kT$ and $t=(k+1)T$, k integer, can be obtained by a rotation of the shown trajectories through an angle $k\Phi$ around the origin.

In order to make the periodicity more apparent, the same orbits are represented in Figure 2 in the rotating system of axes (X, Y). In this system, the orbits are closed. It becomes apparent also that the orbits have an additional symmetry with respect to the Y-axis, provided that bodies 1 and 3 are exchanged as the symmetry is effected. This is a consequence of our particular choice $m_1 = m_3$; it is analogous to the particular symmetry exhibited by the restricted problem when the masses of the primaries are equal (see Szebehely, 1967). This symmetry, together with the symmetry with respect to the X-axis, might be used to reduce the integration to one fourth of

TABLE I

Data for family m

Orbit number	A	x_1	v_1	x_2	v_2	T	Φ	k_1	k_2
1	0.	−0.717 560 83	∞	−0.717 560 83	−∞	4.698 083 80	0.	−1.988 976	0.062 601
2	0.05	−0.707 716 75	6.328 009 73	−0.715 942 47	−6.373 788 14	4.700 788 51	0.036 312 27	−1.966 584	0.095 367
3	0.10	−0.675 216 05	2.986 507 53	−0.710 740 24	−3.078 989 56	4.707 639 64	0.067 383 12	−1.797 152	0.182 801
4	0.12	−0.653 074 33	2.385 677 68	−0.707 335 46	−2.497 176 15	4.709 672 73	0.073 651 42	−1.585 142	0.224 118
5	0.123 378 55	−0.648 558 04	2.299 666 66	−0.706 656 20	−2.414 382 52	4.709 739 12	0.073 834 10	−1.530 925	0.230 658
6	0.13	−0.638 850 75	2.140 088 21	−0.705 214 50	−2.261 094 50	4.709 413 09	0.072 984 16	−1.400 727	0.242 462
7	0.14	−0.621 388 22	1.915 512 98	−0.702 685 97	−2.045 904 87	4.706 991 47	0.067 049 02	−1.116 143	0.255 641
8	0.15	−0.598 417 94	1.698 383 01	−0.699 491 39	−1.837 733 41	4.699 486 05	0.049 891 95	−0.632 671	0.256 177
9	0.155	−0.582 882 12	1.584 329 26	−0.697 416 08	−1.727 697 04	4.691 102 21	0.031 595 61	−0.228 560	0.245 027
10	0.158 614 12	−0.567 855 11	1.491 823 13	−0.695 472 38	−1.637 555 21	4.680 100 18	0.008 233 45	0.225 044	0.225 044
11	0.159 484 57	−0.563 306 30	1.466 654 18	−0.694 895 89	−1.612 810 37	4.676 171 28	0.	0.374 696	0.217 192
12	0.16	−0.560 329 27	1.450 816 33	−0.694 521 40	−1.597 177 23	4.673 444 46	−0.005 689 85	0.475 734	0.211 599
13	0.162	−0.544 995 64	1.376 384 39	−0.692 627 12	−1.522 895 13	4.657 406 99	−0.038 868 46	1.034 184	0.177 183
14	0.163	−0.528 533 51	1.307 922 52	−0.690 652 38	−1.452 940 59	4.636 415 08	−0.081 899 78	1.700 334	0.130 125
15	0.163 092 67	−0.521 581 13	1.282 039 76	−0.689 834 94	−1.425 929 19	4.626 381 17	−0.102 411 16	2.	0.107 296
16	0.163	−0.514 500 99	1.257 323 92	−0.689 011 77	−1.399 770 38	4.615 466 37	−0.124 723 44	2.314 724	0.082 365
17	0.162	−0.496 638 59	1.201 531 74	−0.686 974 01	−1.339 085 36	4.584 950 97	−0.187 290 24	3.141 449	0.012 634
18	0.16	−0.478 383 06	1.152 869 53	−0.684 947 23	−1.283 725 66	4.549 762 30	−0.260 120 16	4.010 024	−0.067 174
19	0.155	−0.447 766 03	1.086 414 82	−0.681 692 75	−1.202 837 18	4.483 471 90	−0.400 347 25	5.417 933	−0.214 794
20	0.15	−0.422 945 39	1.043 465 06	−0.679 240 85	−1.145 947 30	4.425 000 22	−0.528 145 37	6.405 341	−0.342 288
21	0.14	−0.376 725 42	0.982 246 60	−0.675 430 62	−1.055 526 77	4.310 221 43	−0.792 162 67	7.579 348	−0.588 814

22	0.135	−0.352 673 26	0.957 526 21	−0.674 034 19	−1.014 729 78	4.249 185 92	−0.940 175 59	7.763 579	−0.720 133
23	0.13	−0.326 401 91	0.934 675 28	−0.673 144 89	−0.974 077 58	4.182 235 69	−1.108 685 36	7.615 196	−0.865 884
24	0.125	−0.295 878 95	0.912 470 77	−0.673 151 72	−0.931 195 63	4.104 351 41	−1.312 458 19	7.019 772	−1.038 365
25	0.12	−0.255 884 00	0.888 907 04	−0.675 262 14	−0.880 946 31	4.002 058 00	−1.591 201 75	5.692 509	−1.267 800
26	0.118	−0.233 130 60	0.877 758 83	−0.677 726 93	−0.854 940 81	3.943 533 06	−1.755 214 10	4.752 062	−1.397 642
27	0.116	−0.193 116 68	0.861 379 31	−0.684 668 05	−0.813 114 05	3.839 475 95	−2.052 095 54	2.969 901	−1.615 567
28	0.115 713 76	−0.170 963 74	0.853 831 43	−0.690 107 61	−0.791 865 62	3.780 993 49	−2.220 426 34	2.	−1.724 356
29	0.116	−0.148 550 84	0.847 156 59	−0.696 883 57	−0.771 605 58	3.721 042 48	−2.392 982 36	1.086 557	−1.820 454
30	0.118	−0.106 719 97	0.836 982 46	−0.713 236 16	−0.736 771 02	3.606 824 85	−2.718 812 58	−0.319 573	−1.947 506
31	0.12	−0.082 151 62	0.832 233 39	−0.725 241 76	−0.717 919 76	3.538 330 45	−2.910 752 03	−0.914 802	−1.984 752
32	0.125	−0.037 658 98	0.825 617 24	−0.751 774 38	−0.686 384 88	3.412 088 65	−3.254 684 18	−1.495 139	−1.998 805
33	0.13	−0.002 769 91	0.821 965 99	−0.777 026 87	−0.663 686 74	3.311 924 41	−3.516 728 52	−1.525 269	−1.970 533
34	0.135	0.027 607 80	0.819 704 31	−0.802 220 47	−0.645 168 52	3.224 734 42	−3.736 179 69	−1.376 257	−1.850 643
35	0.14	0.055 313 37	0.818 264 36	−0.827 764 02	−0.629 170 51	3.145 936 52	−3.927 277 77	−1.170 591	−1.648 703
36	0.145	0.081 260 11	0.817 362 75	−0.853 844 84	−0.614 882 45	3.073 338 98	−4.097 151 42	−0.943 993	−1.399 690
37	0.15	0.105 980 53	0.816 832 88	−0.880 562 60	−0.601 842 09	3.005 711 87	−4.250 024 60	−0.711 918	−1.130 193
38	0.16	0.153 007 80	0.816 488 79	−0.936 116 92	−0.578 440 90	2.882 552 58	−4.515 146 02	−0.262 848	−0.592 379
39	0.18	0.242 172 71	0.817 160 07	−1.056 305 60	−0.538 581 21	2.674 161 03	−4.925 096 98	0.484 762	0.303 561
40	0.20	0.329 490 34	0.818 243 51	−1.188 922 24	−0.504 610 61	2.505 242 79	−5.222 253 14	1.005 753	0.910 823
41	0.25	0.555 308 40	0.819 269 71	−1.574 218 98	−0.435 695 63	2.204 721 73	−5.673 359 08	1.644 458	1.624 717
42	0.3	0.803 874 70	0.817 314 45	−2.033 949 89	−0.382 454 11	2.016 097 91	−6.093 675 02	1.859 07	1.854 22
43	0.4	1.390 215 10	0.808 870 76	−3.168 086 74	−0.306 020 38	1.806 574 23	−6.108 685 63	1.9697	1.9692
44	0.5	2.105 927 47	0.799 195 32	−4.580 982 64	−0.254 392 79	1.700 090 25	−6.188 776 03	1.9911	1.9911
45	0.6	2.954 838 48	0.790 324 79	−6.268 440 65	−0.217 442 13	1.638 746 48	−6.226 339 96	1.9968	1.9968
46	0.7	3.938 098 29	0.782 642 84	−8.228 559 03	−0.189 773 16	1.600 148 59	−6.246 281 52	1.9986	1.9986
...									
	∞	∞	0.707 106 78	$-\infty$	0.	1.480 960 98	−6.283 185 31	2.	2.

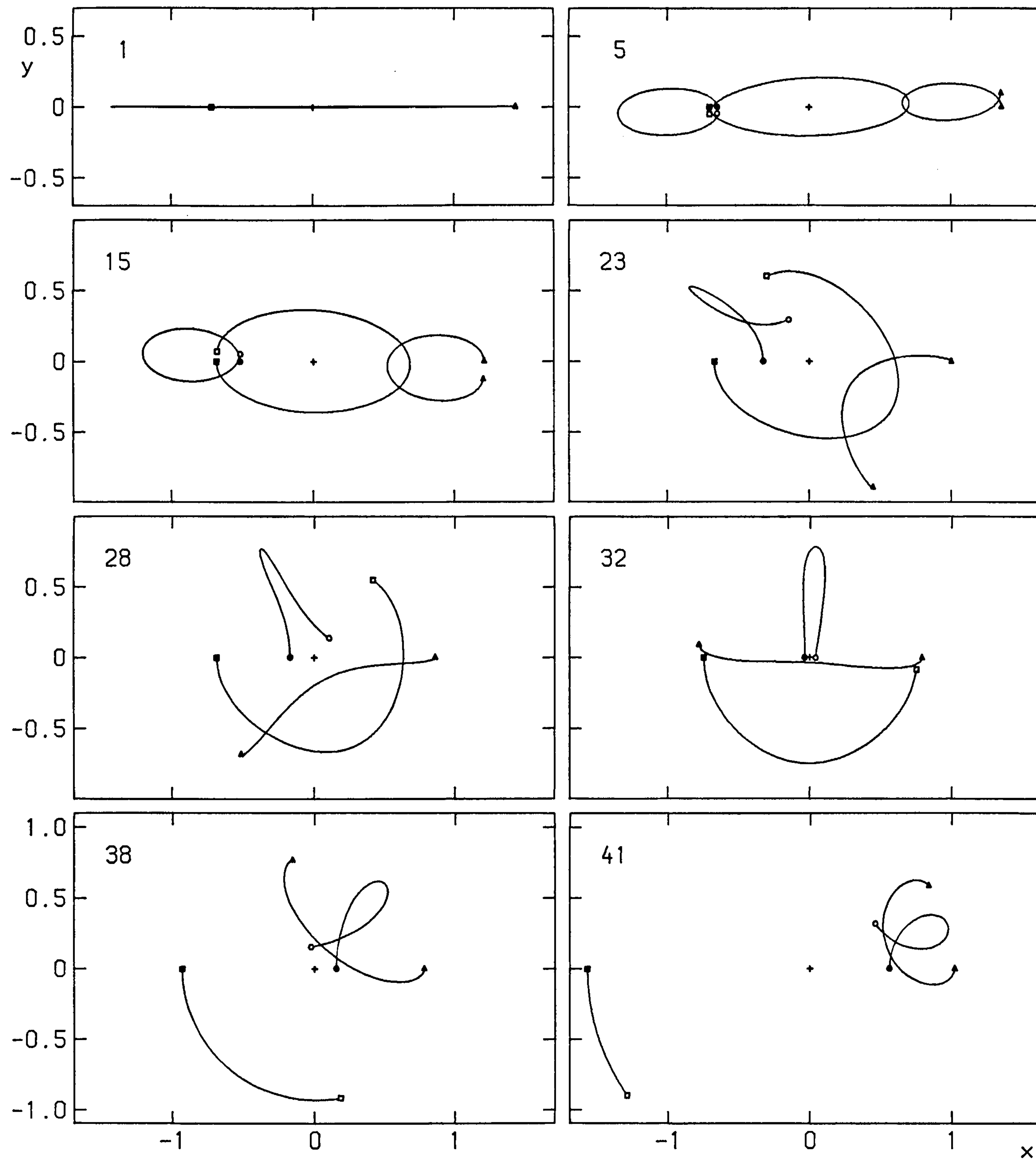

Fig. 1. Some orbits of family m, in the fixed system of axes (x, y). One full period is represented. Filled symbols represent positions at time $t=0$; open symbols are positions at time $t=T$. Circles, squares and triangles correspond respectively to bodies 1, 2, 3. The cross indicates the centre of mass.

the whole period; such a reduction was indeed used in Schubart's original computation (1956).

We proceed now to a more detailed description of the orbits and their evolution along the family. Orbit 1 is Schubart's rectilinear orbit; it is the same in Figures 1 and 2, since the rotation angle is zero for that particular orbit. As we move a little along the family, the orbits begin to extend in the y-direction (cf. orbit 5). Body 2 acts as an intermediary between the two others: at time $t=0$ there is a close approach of bodies 1 and 2, while at time $t=T/2$ there is a close approach of bodies 2 and 3. Thus, body 2 periodically kicks away bodies 1 and 3 in turn, and prevents them from ever approaching the centre. This kind of motion has been called *interplay* by Szebehely (1971).

The angle of rotation Φ is at first positive and small; it reaches a maximum, then decreases. The periodic orbit corresponding to the maximum was located numerically;

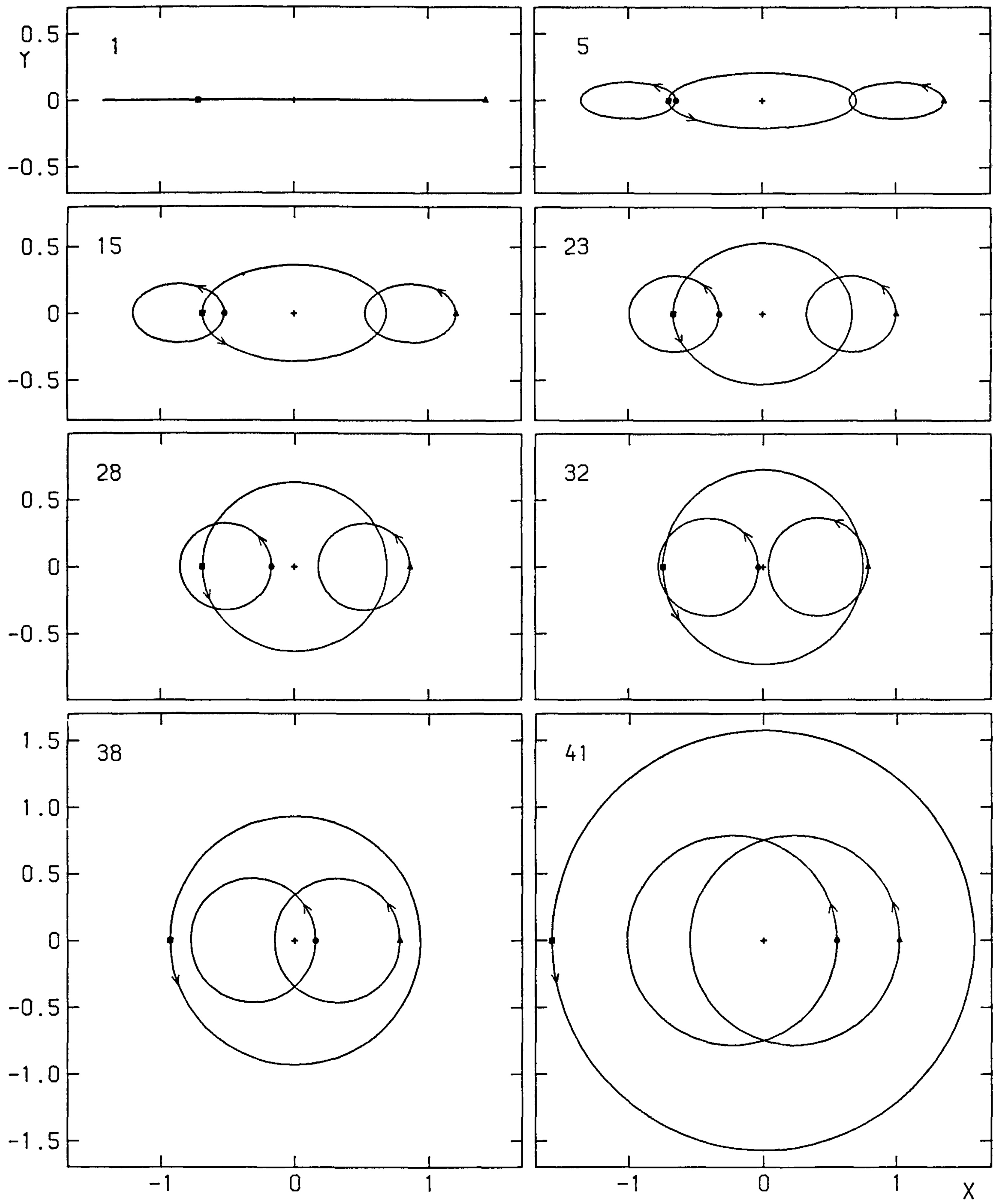

Fig. 2. Same orbits as in Figure 1, represented in a rotating system. Arrows indicate the direction of motion.

it is orbit 5. We noticed then the curious fact that this orbit appeared to correspond also to a maximum for the period T. This observation led us to the discovery of the following general relation:

$$\frac{\mathrm{d}T}{\mathrm{d}A} = -\frac{A}{2E}\frac{\mathrm{d}\Phi}{\mathrm{d}A}. \tag{10}$$

The derivatives with respect to A are to be understood as taken along the family of periodic orbits. A proof of (10) will be given elsewhere (Hénon, 1975). This relation shows that an extremum of Φ must also be an extremum of T. Conversely, an extremum of T is either an extremum of Φ, or an orbit with zero angular momentum; the second case is realized in the present family by Schubart's orbit, for which $A=0$, $\mathrm{d}T/\mathrm{d}A=0$, $\mathrm{d}\Phi/\mathrm{d}A\neq 0$. It should be noted that the relation (10) is true only when all periodic orbits are normalized to the same energy E.

After its maximum, the rotation angle Φ decreases, and later becomes negative. We have computed the particular orbit for which $\Phi=0$; this is orbit 11. This orbit is closed also in fixed axes. It was also found by Broucke (1975), who calls it an 'absolute periodic solution'. Φ continues then to decrease until the end of the family, and tends asymptotically towards -2π. Thus, the orbit of body 2 progressively becomes shorter (Figure 1, orbits 23 to 41). At the same time, bodies 1 and 3 progressively approach each other. Towards the end (orbit 41), body 2 has been pushed aside and bodies 1 and 3 are engaged in close and continuous interaction. We recognize here another familiar kind of motion: bodies 1 and 3 form a close binary, rotating in the retrograde direction; and this binary, considered as a single body, forms itself a wide binary with body 2, rotating in the direct sense. This motion has been called 'revolution' by Szebehely (1971); we shall call it more specifically *retrograde revolution* in order to

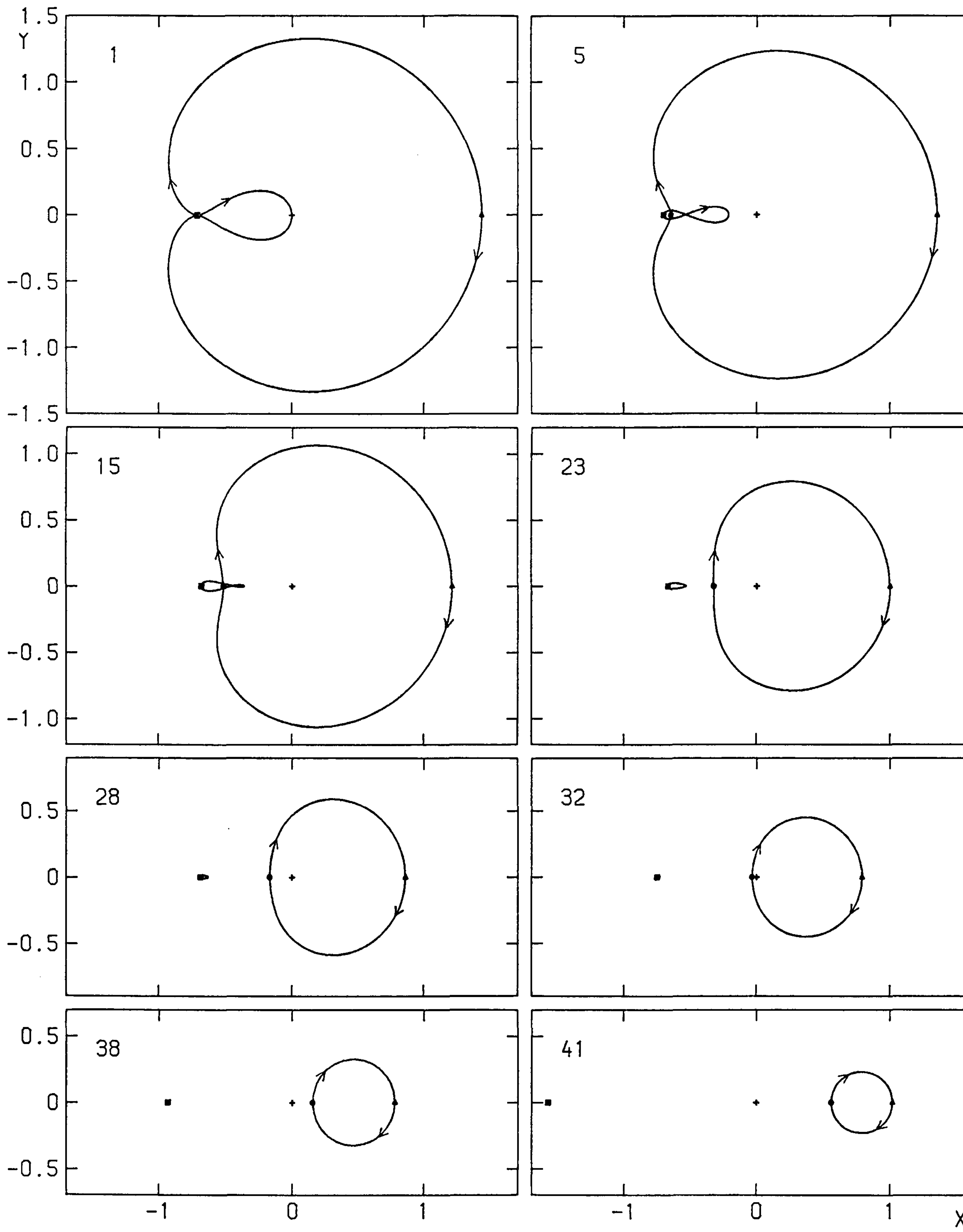

Fig. 3. Same orbits as in Figure 1, represented in another rotating system.

indicate the fact that the binaries revolve in opposite directions. We shall also adopt Harrington's (1969) terminology and speak of the *inner binary* and the *outer binary*.

As already pointed out in I, the angle Φ is defined only modulo 2π. The values given in Table I were defined by continuity, taking $\Phi=0$ for Schubart's orbit. These values have been used to define the rotating system in Figure 2. Near the end of the family, however, Φ approaches -2π, and a better choice is $\Phi'=\Phi+2\pi$. In Figure 3, the orbits are shown in a system of axes rotating with the constant angular velocity $\omega'=\Phi'/T$. The motion becomes very simple in this system towards the end of the family: body 2 is almost motionless, while bodies 1 and 3 revolve around each other in quasi-circular orbits. Incidentally, because of $m_1=m_3$, the orbits of bodies 1 and 3 are identical in Figure 3, and body 2 describes twice the same curve during one period; after a time $T/2$, the motion reproduces itself with bodies 1 and 3 exchanged (cf. Broucke and Boggs, 1975).

The end of the family is now clear. The ratio of the semi-major axis of the outer binary to that of the inner binary increases constantly and tends to infinity. The energy of the outer binary becomes negligible compared to that of the inner binary; since the orbits are normalized to a constant total energy (2), the semi-major axis of the inner binary tends to the limiting value $\frac{1}{3}$, and the semi-major axis of the outer binary tends to infinity. Inversely, the angular momentum of the system becomes essentially the angular momentum of the outer binary, and tends to $+\infty$. The period of the system becomes essentially the period of the inner binary, and tends to $\pi\sqrt{\frac{2}{3}}$. The velocities v_1 and v_3 at $t=0$ tend to $\pm 1/\sqrt{2}$, while v_2 tends to zero.

The family can also be extended, starting from Schubart's orbit, towards negative values of A. The corresponding orbits are obtained from those of Figure 1 by a symmetry with respect to the x-axis, and the parameters are obtained by reversing the signs of A, v_1, v_2, Φ in Table I. We reach thus the other end of the family, which is again a retrograde revolution, but this time with the inner binary revolving in the direct sense and the outer binary revolving in the retrograde sense; the angular momentum tends to $-\infty$.

3. Stability: Method

The planar three-body problem in its original form has 6 degrees of freedom. However, there are 4 time-independent integrals: the two components u_0, v_0 of the velocity of the centre of mass; the total energy E; and the angular momentum A. Four ignorable coordinates are associated to these integrals: the coordinates x_0, y_0 of the centre of mass; the time t; and an angle θ defining the orientation of the figure (θ can be defined, for instance, as the angle between the line joining body 2 to body 3 and the x-axis). As a consequence, the number of degrees of freedom can be reduced from 6 to 2 (Whittaker, 1937, pages 54 and 64); the 4 integrals appear as parameters in the reduced system, while the ignorable coordinates are altogether eliminated. The reduction can be carried out in many different ways (see for instance Hadjidemetriou, 1975; Broucke, 1975). Here we shall consider a rotating system of axes $(\hat{x}, \hat{y})$, with its origin at the centre of mass and with the $\hat{x}$-axis parallel at any given time to the direction from 2 to 3. We take the coordinates of position and velocity of body 1: $\hat{x}_1, \hat{y}_1, \hat{u}_1, \hat{v}_1$, as the four dependent variables of the reduced system; $\hat{u}_1$ and $\hat{v}_1$ are the components along the rotating axes of the velocity measured in *fixed* axes. We also take $s = \dot{r}_{23}$, i.e. the time-derivative of the distance from 2 to 3, as independent variable. The equations of the reduced system are then of the form

$$\frac{\mathrm{d}\hat{x}_1}{\mathrm{d}s} = f(s, \hat{x}_1, \hat{y}_1, \hat{u}_1, \hat{v}_1, u_0, v_0, E, A) \tag{11}$$

and three similar equations for $\mathrm{d}\hat{y}_1/\mathrm{d}s$, $\mathrm{d}\hat{u}_1/\mathrm{d}s$, $\mathrm{d}\hat{v}_1/\mathrm{d}s$. A straightforward computation shows that it is indeed possible to reduce the system to this form.

The integrals u_0, v_0, E, A appear as parameters in the Equations (11). Changing u_0, v_0 amounts merely to changing the system of reference, and we can take $u_0 = v_0 = 0$ without loss of generality. Similarly, E can be reduced to the normalized value (2) by a change of scale. Thus the Equations (11) contain in fact only one non-trivial parameter, A.

It is customary to speak of 'periodic solutions of the three-body problem'; strictly speaking, one means in fact periodic solutions of the reduced problem. Corresponding solutions of the original problem will not in general be periodic: the final configuration is rotated and translated with respect to the initial one. So we are in fact investigating periodic solutions of (11).

The new independent variable s does not grow monotonically along the orbit, but oscillates. For given values of the integrals u_0, v_0, E, A, we define a particular solution of the reduced system by the values of $\hat{x}_1, \hat{y}_1, \hat{u}_1, \hat{v}_1$ for $s=0$; the whole solution can then be obtained by integration of (11). If, at a later passage of s through the value 0, $\hat{x}_1 \ldots \hat{v}_1$ take again the same values, the orbit is periodic. Thus, periodic solutions of the reduced system must satisfy 4 equations for 4 unknowns, and therefore they are isolated.

In order to determine the stability of a periodic orbit, we consider a slightly perturbed orbit starting from $\hat{x}_1 + \Delta\hat{x}_1 \ldots \hat{v}_1 + \Delta\hat{v}_1$ at $s=0$. The values of the parameters u_0, v_0, E, A are kept fixed. We follow this perturbed orbit until s again becomes zero. The four variables are then $\hat{x}_1 + \Delta'\hat{x}_1 \ldots \hat{v}_1 + \Delta'\hat{v}_1$. In the linear approximation, there is

$$\begin{pmatrix} \Delta'\hat{x}_1 \\ \Delta'\hat{y}_1 \\ \Delta'\hat{u}_1 \\ \Delta'\hat{v}_1 \end{pmatrix} = \mathbf{R} \begin{pmatrix} \Delta\hat{x}_1 \\ \Delta\hat{y}_1 \\ \Delta\hat{u}_1 \\ \Delta\hat{v}_1 \end{pmatrix} \tag{12}$$

where $\mathbf{R}$ is a 4×4 matrix. The 4 eigenvalues of $\mathbf{R}$ are those we need.

In practice, the reduced Equations (11) are awkward to use for numerical integration, and it is much more convenient to integrate the original unreduced equations. To any given solution of the reduced system (11), there corresponds a four-fold infinity of solutions of the original system, because an arbitrary constant can be added to each of the 4 ignorable coordinates x_0, y_0, t, θ. The choice of these constants has no effects on the results. We therefore set simply $x_0 = y_0 = t = \theta = 0$ for the initial point $s = 0$. The study of the stability of a periodic solution is then very similar to the search for the periodic orbit itself; in fact, the matrix **R** is precisely the 4×4 matrix which was already defined and used in I.

Since the three-body problem is a Hamiltonian system, if λ is an eigenvalue, then λ^{-1} is also an eigenvalue (Whittaker, 1937, page 403). Therefore the equation for the eigenvalues has the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_1\lambda + 1 = 0. \tag{13}$$

An exhaustive study of this equation has been made by Broucke (1969), who showed that seven regions must be distinguished in the (a_1, a_2) plane. Only one region corresponds to the stability of the periodic orbit; it is defined by

$$-4 < a_1 < 4, \qquad 2|a_1| - 2 < a_2 < \frac{a_1^2}{4} + 2. \tag{14}$$

Equation (13) can also be written

$$(\lambda^2 - k_1\lambda + 1)(\lambda^2 - k_2\lambda + 1) = 0, \tag{15}$$

where k_1 and k_2 are given by

$$k_{1,2} = \tfrac{1}{2}[-a_1 \pm (a_1^2 - 4a_2 + 8)^{1/2}]. \tag{16}$$

k_1 and k_2 are the *stability indices* of the orbit (Whitaker, 1937, page 404; Broucke, 1969). The orbit is stable if

$$k_1 \text{ and } k_2 \text{ real}, \qquad |k_1| < 2, \qquad |k_2| < 2. \tag{17}$$

There is some question as to which quantities are most convenient for a description of the stability. a_1 and a_2 have the advantage that they are always real, so that a periodic orbit can always be represented by a point in the (a_1, a_2) plane; on the contrary, k_1 and k_2 can be complex, in which case the orbit cannot be represented in the (k_1, k_2) plane. On the other hand, the region of stability (14) in the (a_1, a_2) plane has a complicated shape, while in the (k_1, k_2) plane it is simply a square; and the conditions (17) are a natural generalization of the case of two degrees of freedom, where one has a single condition of the form $|k| < 2$. Moreover, in some cases the perturbation Equations (12) are separable and the two stability indices k_1 and k_2 can be related to two different kinds of perturbations (Hénon, 1973; see also below). For the orbits considered in the present paper, k_1 and k_2 happen to be always real, and we shall use them.

In passing, we point out an error in a similar analysis made by Bray and Goudas (1967). They state that the periodic orbit is stable when $a_1^2 - 4a_2 + 8 < 0$. This is incorrect; in that case, which corresponds to Broucke's region 2, the four eigenvalues are complex and two of them have a modulus larger than 1.

The practical procedure is as follows. After a periodic orbit has been found, we again compute 4 slightly perturbed orbits in which each of the initial coordinates in turn is increased by a small quantity ε. For instance, starting from $\hat{x}_1+\varepsilon$, $\hat{y}_1$, $\hat{u}_1$, $\hat{v}_1$, we find the final values $\hat{x}_1+\Delta'\hat{x}_1 \ldots \hat{v}_1+\Delta'\hat{v}_1$. For increased accuracy, we also compute symmetrically perturbed orbits: starting from $\hat{x}_1-\varepsilon$, $\hat{y}_1$, $\hat{u}_1$, $\hat{v}_1$, we obtain the final values $\hat{x}_1+\Delta''\hat{x}_1 \ldots \hat{v}_1+\Delta''\hat{v}_1$. The 16 elements of $\mathbf{R}$ are then obtained from

$$r_{11} = \frac{\Delta'\hat{x}_1 - \Delta''\hat{x}_1}{2\varepsilon}, \quad \text{etc.} \tag{18}$$

Next, we compute the coefficients $a_1 \ldots a_4$ of the eigenvalue equation

$$\mathrm{Det}\,(\mathbf{R} - \lambda\mathbf{I}) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \tag{19}$$

There should be $a_3=a_1$ and $a_4=1$; this serves as a check on the accuracy of the computation. With $\varepsilon=10^{-6}$ and 16-digit computer accuracy, these conditions are usually satisfied with an error of the order of 10^{-8} for the present family m. Internal evidence indicates that a_1 is more accurately determined than a_3; therefore only a_1 and a_2 are retained. Finally, k_1 and k_2 are computed from (16).

Hadjidemetriou (1975) has developed a similar method for the determination of the stability. His system of reduced variables is not the same as the one used here; this is of no consequence, because the eigenvalues are intrinsic properties of a periodic orbit and do not depend on the variables used to describe it. We have verified this by recomputing Hadjidemetriou's orbits and their stability indices in our own system. Our method is applicable to any plane periodic orbit; Hadjidemetriou's method is presented only for symmetric orbits, but it could probably be generalized also to non-symmetric periodic orbits.

4. Stability: Results

The method described in the previous Section was used to determine the stability of the orbits of family m, itself described in Section 2. The stability indices k_1 and k_2

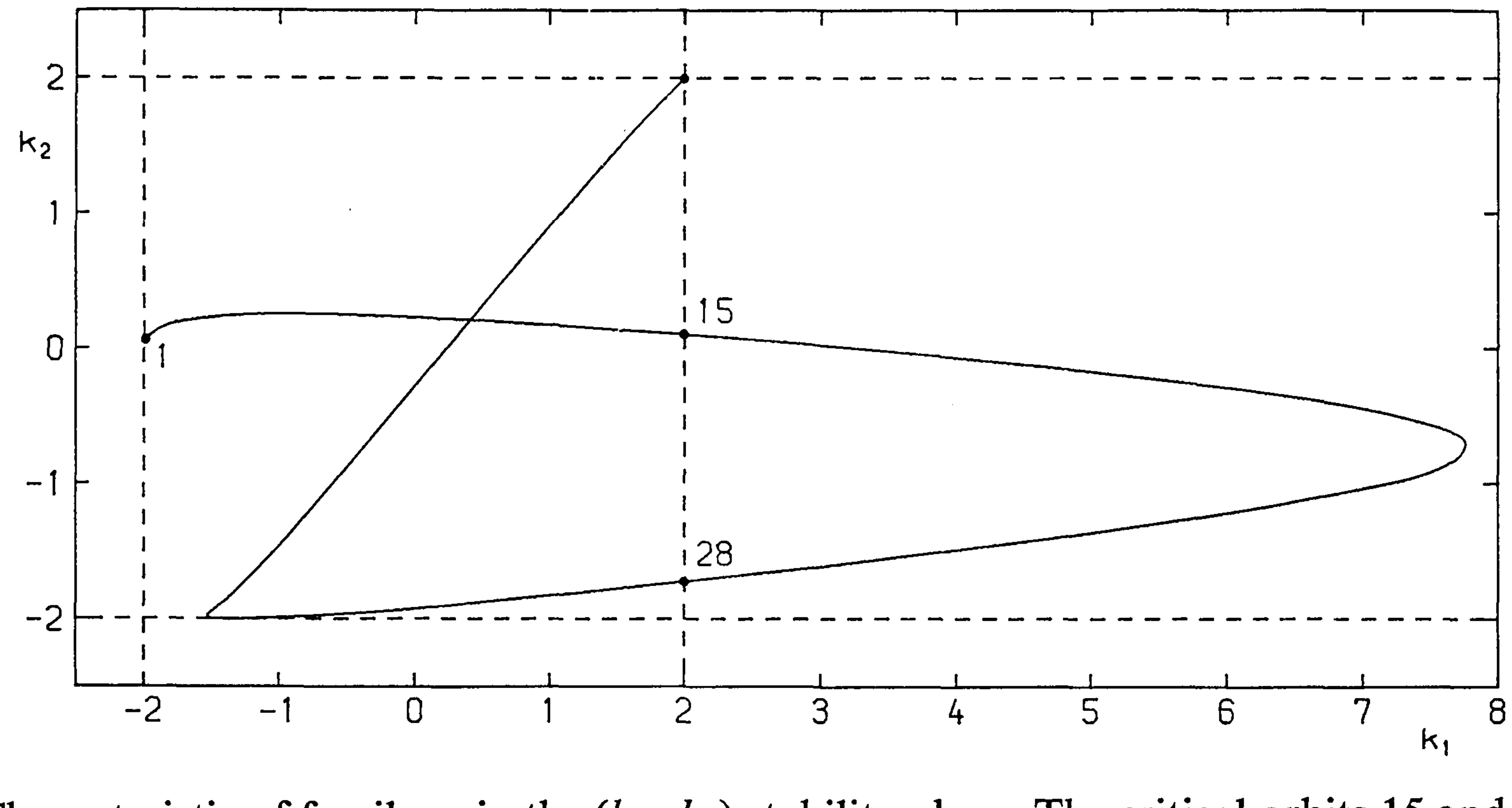

Fig. 4. Characteristic of family m in the (k_1, k_2) stability plane. The critical orbits 15 and 28 and the end orbits of the family are represented by dots.

are listed in Table I. Figure 4 represents the (k_1, k_2) plane; each periodic orbit is represented by a point, and the whole family is represented by a curve.

We notice first the remarkable fact that Schubart's rectilinear orbit (orbit 1) is stable, in spite of its collisions: both stability indices are less than 2 in absolute value. $|k_1|$ is close to, but definitely less than 2. For that particular orbit, the Equations (12) separate into two independent systems:

$$\begin{pmatrix} \Delta'\hat{x}_1 \\ \Delta'\hat{u}_1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{13} \\ r_{31} & r_{33} \end{pmatrix} \begin{pmatrix} \Delta\hat{x}_1 \\ \Delta\hat{u}_1 \end{pmatrix}, \qquad \begin{pmatrix} \Delta'\hat{y}_1 \\ \Delta'v_1 \end{pmatrix} = \begin{pmatrix} r_{22} & r_{24} \\ r_{42} & r_{44} \end{pmatrix} \begin{pmatrix} \Delta\hat{y}_1 \\ \Delta v_1 \end{pmatrix}. \tag{20}$$

The first system corresponds to perturbations along the $\hat{x}$-axis, i.e. perturbations which preserve the rectilinear nature of the orbit; the second system corresponds to perturbations perpendicular to the $\hat{x}$-axis. This is rather analogous to the separation between horizontal and vertical stability for plane orbits (Hénon, 1973; see also Section 5). As a consequence, specific meanings can be assigned to k_1 and k_2; we choose to define k_1 as the index for longitudinal stability and k_2 as the index for transversal stability. Then:

$$k_1 = r_{11} + r_{33}, \qquad k_2 = r_{22} + r_{44}. \tag{21}$$

When one moves away from Schubart's orbit along the family, however, the separation property disappears. k_1 and k_2 are then the two roots of a second-degree equation, as shown by (16), and cannot be distinguished any more. In Table I, the definition of k_1 and k_2 has been made simply by continuity.

As family m is followed, we see from Table I and Figure 4 that the orbits continue to be stable over a sizable interval. This fact has important consequences, which will be developed in Section 5. Then, at some point, k_1 becomes larger than 2 and the orbits become unstable. The orbit for which k_1 is exactly 2 is of particular interest; this is orbit 15 in Table I. We call it a *critical orbit*, in analogy with the similar situation in the restricted problem. It is represented in the third frame of Figures 1 to 3.

The numerical results indicate that this critical orbit corresponds exactly to an extremum of A along the family. This is similar to the result found in the restricted problem (Hénon, 1965) that an extremum of the Jacobi constant C corresponds to a critical orbit. We shall not give here a rigorous proof of the fact, but only the following intuitive explanation. Consider a periodic orbit which corresponds to an extremum of A, and an infinitesimal perturbation which changes it into a neighbouring periodic orbit of the same family. This perturbation does not change A to first order since we are at an extremum. It is therefore a valid perturbation of the reduced system (see Section 3). On the other hand, the perturbation after one period will be identical to the initial perturbation. It corresponds therefore to an eigenvector of the matrix $\mathbf{R}$, associated to an eigenvalue 1. Then (15) shows that either k_1 or k_2 is equal to 2. Hence our result: *an extremum of A corresponds to a critical orbit*. It should be noted that, here again, the property holds because all orbits are normalized to the same energy.

After the critical orbit 15, the orbits of family m become unstable, and remain so until a second critical orbit is reached: this is orbit 28 in Table I. Again it corresponds to an extremum of A. It is represented in the fifth frame of Figures 1 to 3. After that orbit, both k_1 and k_2 are less than 2 in absolute value, and the orbits are stable again. In the vicinity of orbit 32, k_2 reaches a minimum value equal to $-1.998\,992\ldots$, slightly but definitely above the critical value -2: the curve in Figure 4 seems to touch the critical line $k_2=-2$, but in reality it does not. The orbits remain stable until the end of the family. This stability of the revolution case is not surprising, since the system is practically decoupled into two independent two-body systems (cf. Harrington, 1968, 1969, 1972). In this limit, k_1 and k_2 can again be given separate meanings: an examination of the detailed results shows that k_1 corresponds to perturbations of the outer binary, while k_2 corresponds to perturbations of the inner binary. Using the two-body approximation, one can show that

$$k_1 \approx k_2 \approx 2\cos\Phi. \tag{22}$$

As the limit is approached, Φ tends to -2π, and k_1 and k_2 tend to 2 from below. The accuracy of the numerical values of k_1 and k_2 in Table I deteriorates somewhat towards the end of the family, because the quantity under the square root in (16) tends to zero.

TABLE II

Stability indices of family 1

A	k_1	k_2
0.	19.037 599	5.528 887
0.001	18.959 247	5.838 942
0.002	18.847 851	6.937 154
0.003	18.925 605	8.918 414
0.004	19.442 487	11.428 346
0.005	20.926 603	13.713 341
0.006	23.948 034	15.044 280
0.007	28.245 930	15.588 801

TABLE III

Stability indices of family 2

A	k_1	k_2
0.	3.301 132	−66.351 928
0.005	3.298 900	−65.812 396
0.010	3.290 721	−64.106 512
0.015	3.271 190	−60.922 853
0.020	3.225 735	−55.482 264
0.025	3.095 866	−45.250 319

Tables II and III give the values of k_1 and k_2 (names arbitrarily assigned) for the orbits of the two families described in Paper I. Both families are strongly unstable, with large values of $|k_1|$ and $|k_2|$.

5. Comments

(a) 'Interplay' orbits in general seem to have the property that one of the bodies acts as an intermediary between the two other bodies and prevents them from approaching each other (Szebehely, 1971). Let us call body 2 the intermediary. An interplay periodic orbit can then be loosely characterized by the numbers n_{12} and n_{23} of close approaches (or collisions) occurring respectively between bodies 1 and 2 and between bodies 2 and 3 during one period. (This is a generalization of the classification proposed by Szebehely, 1970.) There must be at least one close approach of each kind for the motion to be classified as interplay; therefore the simplest possible case is $(n_{12}, n_{23})=(1, 1)$. This is precisely the case of family m (see Figure 1, orbits 1, 5, 15). Therefore this family may be said to represent the simplest possible kind of interplay. For comparison, we note that the other known periodic orbits of the interplay type, described by Szebehely and Peters (1967), Standish (1970), Szebehely (1970), Szebehely and Feagin (1973), correspond to the various cases (9, 8), (5, 4), (4, 4), (4, 3), and (3, 2).

(b) We have described the family m for the case of three equal masses. One may ask how the family will evolve when the masses are changed. It is known that periodic orbits of the 'retrograde revolution' type exist and form a one-parameter family for any given masses of the three bodies (Siegel and Moser, 1971, page 138). Therefore, if the mass m_2 is decreased continuously from $\frac{1}{3}$ to 0, while the other masses m_1 and m_3 are kept finite, the revolution end of family m will continuously evolve, until for $m_2=0$ we have a massless body describing a quasi-circular orbit of large radius, in the retrograde direction, around two bodies of finite mass in direct circular motion around their common centre of mass. We recognize here family m of periodic orbits of the restricted problem of three bodies (Strömgren, 1933). If, instead of m_2, we let the mass of m_1 or m_3 tend to zero, our family will again evolve continuously, and this time will end in family f or h of the restricted problem. Schubart (1956) had already noted the connection between his rectilinear orbits and Strömgren's family f.

Thus, if the masses are considered as variable, we have a three-parameter family of periodic orbits of the kind considered in I. This larger family contains in particular the families m, f, h of the restricted problem, for all values of the mass ratio of the primaries, and the family described in the present paper. It could be called the *retrograde revolution family*; here we have called it more briefly *family m*, as a natural generalization of Strömgren's notation.

(c) Hadjidemetriou (1975) and Broucke (1975) have independently computed in part another family of periodic orbits, also for the case of three equal masses. This family begins with direct revolution: bodies 1 and 3 form a close binary, around which body 2 rotates at a larger distance and in the same direction. The theoretical existence proof (Siegel and Moser, 1971, page 138) applies also to this case. Therefore, if the mass m_2 is decreased to zero, this family will evolve into family l of the restricted problem; if m_1 or m_3 is decreased to zero, the family will evolve into family g or i of the restricted problem. We propose to call it 'direct revolution family' or 'family l' in general.

(d) We comment now on the stability of the orbits. A rather unexpected result of our study was that orbits of family m are stable in the interval which extends from orbit 1 to orbit 15 (see Table I and Figure 1). Thus, *there exist stable periodic orbits with the three masses of the same order* (they are in fact equal here) *and with the three distances of the same order*. This result has interesting consequences. The fundamental theorems of Arnold (1963) and Moser (1962) state that in a dynamical system with two degrees of freedom, a linearly stable periodic orbit is surrounded in general by a finite region of phase space in which non-linear stability also is guaranteed: orbits started inside that region will never leave the vicinity of the periodic orbit. Although a similar proof is not available for systems with three or more degrees of freedom (cf. Siegel and Moser, 1971, page 277), it appears very likely that a similar situation exists. Therefore, the stable periodic orbits which we have found are very probably surrounded by a region of finite measure in phase space in which the orbits possess non-linear stability. This means in particular that for initial conditions taken inside that region, *none of the three bodies will ever escape*.

Birkhoff (1927) conjectured that, except for a set of initial conditions of measure zero, triple systems do not remain bound forever, but end with the escape of one of the three bodies. This conjecture, however, seemed to be contradicted in two cases: first, in the vicinity of triangular equilibrium solutions, which are linearly stable when one of the masses is much larger than the other two (Siegel and Moser, 1971, page 120); second, in the 'revolution' case, in which one of the distances is much smaller than the other two. In order to exclude these troublesome cases, Szebehely (1973, 1974) stated a weaker conjecture: if the three masses are of the same order and if the three distances are of the same order, then a triple system cannot remain bounded forever. This conjecture seemed to be borne out by the results of extensive numerical tests (Agekyan and Anosova, 1967, 1968; Standish, 1972; Szebehely, 1972). Our present results, however, suggest that even this weaker form of the conjecture is not true.

(e) For practical applications, the above study of the stability in the plane of motion should be supplemented by a study of the 'vertical stability', i.e. the stability with respect to perturbations perpendicular to the plane, as in the case of the restricted problem (Hénon, 1973). The vertical stability can be studied separately and will be characterized by a third stability index, k_3. An orbit which is stable in the plane can be unstable vertically; in fact, this was found to happen precisely for family m in the restricted problem (*ibid.*).

In the particular case of Schubart's rectilinear orbit, however, the answer to the three-dimensional stability problem can be given without any additional computation. This orbit has axial symmetry with respect to the x-axis, and therefore the vertical stability index k_3 is equal to the transversal stability index k_2. Thus, *Schubart's orbit is stable in three-dimensional space.* Moreover, since k_3 varies continuously along the family, there must exist a finite interval around Schubart's orbit where $|k_3| < 2$, and we can predict that *family m contains a finite interval of three-dimensionally stable periodic orbits.*

(f) These results open up the possibility that there exist triple systems of stars with a motion completely different from the usual revolution type. Such systems do not seem to have been observed yet. However, the number of reliably known triple systems is very limited (Harrington, 1972). It might be also that the regions of stability surrounding the stable periodic orbits, although finite, are not very large. Finally, the circumstances of the formation of triple systems might be such that only the stable region corresponding to the revolution case can be populated. Nevertheless, the possibility of a different kind of motion should be kept in mind, particularly when interpreting incomplete data for an observed triple system.

(g) We consider now the interval of stability at the other end of the family, i.e. from orbit 28 to the end. Harrington (1972) computed a number of non-periodic orbits for an equal-mass triple system, starting both from direct and retrograde revolution configurations. In the retrograde revolution case, he concluded that the system is stable (in the sense that there is no significant change of the osculating elements during the period of integration) only when the ratio of the outer periastron distance q_2 to the inner semi-major axis a_1 is greater than 2.75. On the other hand, we find that the periodic orbits become linearly unstable only at the critical orbit 28, for which the above ratio q_2/a_1 is about 1 as can be seen from Figure 3. Thus our results indicate a larger range of stability than those of Harrington. A possible explanation of this discrepancy is as follows: as one approaches the critical orbit, the region of stability surrounding the periodic orbit in phase space presumably shrinks, and vanishes at the critical orbit itself. Harrington did not look specifically

TABLE IV

Osculating elements of inner and outer binaries

Orbit number	t	a_1	e_1	a_2	e_2	q_2/a_1
28	0	1.470 822	0.298 328	1.918 647	0.460 473	0.703 798
	$T/4$	0.733 096	0.611 209	0.880 966	0.078 594	1.107 260
32	0	0.978 754	0.154 954	1.401 439	0.195 354	1.152 140
	$T/4$	0.690 583	0.310 891	1.094 483	0.002 639	1.580 686
38	0	0.664 001	0.051 054	1.489 231	0.057 114	2.114 718
	$T/4$	0.599 439	0.092 311	1.421 451	0.015 912	2.333 570
41	0	0.466 463	0.006 133	2.381 736	0.008 568	5.062 201
	$T/4$	0.461 665	0.009 850	2.374 952	0.005 832	5.114 322

for the periodic orbits or their vicinity; he explored the whole phase space for the three-dimensional problem, and therefore he had to use a grid of rather widely spaced initial points. It might be that the region of stability for small q_2/a_1 was simply missed by these points.

Another relevant consideration is that, as the ratio q_2/a_1 decreases, the mutual perturbations of the two binaries become large and the osculating elements begin to lose their significance. This is illustrated by Table IV, which gives the osculating semi-major axis a_1 and eccentricity e_1 of the inner binary, the similar osculating elements a_2 and e_2 of the outer binary, and the ratio q_2/a_1, at $t=0$ and $t=T/4$, for the last four orbits of Figure 3. For orbit 32 and even more for orbit 28, the osculating elements have large fluctuations and therefore these orbits might have been classified as unstable by Harrington.

There is a similar, although less marked discrepancy in the direct revolution case, where Harrington (1972) finds instability for q_2/a_1 less than 3.5, while the last stable periodic orbit found by Hadjidemetriou (1975) corresponds to $q_2/a_1=2.28$.

(h) In Paper I a normalization was proposed for numerically computed periodic solutions of the three-body problem; in particular, the dimensions of the orbit were fixed by normalizing the total energy E to a constant value (for given masses). The present paper brings to light some unexpected advantages of this normalization: it allows the curious relation between energy, angular momentum, period and rotation angle expressed by (10), and also the property that an extremum of the angular momentum A corresponds to a critical orbit (Section 4). Thus, normalizing all orbits of a family to the same energy appears to be not merely a matter of convenience, but also to reflect the deeper structure of the problem.

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