ON RELATIVE PERIODIC SOLUTIONS OF THE PLANAR GENERAL THREE-BODY PROBLEM*

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Abstract. We describe two relatively simple reductions to order 6 for the planar general three-body problem. We also show that this reduction leads to the distinction between two types of periodic solutions: absolute or relative periodic solutions. An algorithm for obtaining relative periodic solutions using heliocentric coordinates is then described. It is concluded from the periodicity conditions that relative periodic solutions must form families with a single parameter. Finally, two such families have been obtained numerically and are described in some detail.

1. Introduction

In the present article we will discuss two families of relative periodic solutions of the general three-body problem. Absolute periodic solutions are periodic in a fixed coordinate system while relative periodic solutions are periodic in a suitably chosen rotating system (Broucke and Boggs, 1975).

We will also discuss in detail a theoretical property that is intimately related to the existence of such relative periodic solutions. It turns out that the planar general problem can be reduced to a system of order eight by only using the integrals of the center of mass. However, it turns out that we have another remarkable property: the eighth-order system can be reduced to a sixth-order system and a quadrature. This reduction uses only the angular momentum integral and is in fact Jacobi's elimination of the nodes. The relative periodic orbits are periodic solutions of the sixth-order system and the absolute periodic orbits are periodic solutions of a seventh-order system.

The reduction of the equations to order six is well known for a long time; see, for instance, Murnaghan (1936), Van Kampen and Wintner (1937), Lemaître (1952), Deprit and Delie (1961) or Deprit and Roels (1962). However, these authors have failed to note that this reduction leads to a distinction between two types of periodic solutions. It is only in the last year that the distinction between relative and absolute periodic solutions in the general three-body problem has become clear; see Hénon (1974) and Hadjidemetriou (1975) or Broucke and Boggs (1975). For these reasons we include in the present paper two new methods of reduction of the general three-body problem. In the first approach, which is essentially barycentric (Sections 3 through 6), we use the notation of Deprit, Delie and Roels, but we believe that we have a more

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simple derivation because we work directly with the equations of motion rather than the Lagrangian. In this derivation we make explicit use of the angular momentum integral. In the second derivation (Sections 7 through 10), which is essentially heliocentric, we use the Lagrangian and we show that by an appropriate change of variables, a Lagrangian with an ignorable coordinate is obtained. This leads to a Routhian with only three degrees of freedom and the angular momentum integral is obtained as a result of the reduction.

Both reductions that are given are remarkably simple, although the final equations of motion are in all cases more complicated than those of the eighth-order system, for instance, in the heliocentric frame. In all our numerical integrations, we have used the heliocentric equations (35) of Section 7. Another reason for this is that the reduced equations of motion have a singularity at every passage through a collinear configuration (Van Kampen and Wintner, 1937, page 166).

In Section 11, we apply the results of the reductions to discuss relative periodic solutions. In Section 12 we detail a practical algorithm for constructing relative periodic solutions while working with non-rotating heliocentric coordinates. This shows in particular that after all the periodicity conditions have been satisfied, a single parameter remains free; this explains that the periodic solutions form a one-parameter family. Finally, in the last section, we describe our numerical results consisting of two families. Both families were derived from Keplerian limiting cases (Arenstorf, 1967 and 1968). At the moment the end of the two families is not yet known. Several periodic orbits of the two families have binary close approaches and a regularization would be desirable to continue the investigations. We have already started the study of the characteristic exponents of the orbits. These results will be published later.

2. Definition of the Problem

We consider the classical planar general three-body problem with non-zero masses m_0, m_1, and m_2. The gravitation constant is assumed to be unity. In a barycentric inertial frame of reference, the Lagrangian of the system may be written as

$$\mathscr{L} = \tfrac{1}{2} \sum_{i=0}^{2} m_i(\dot{\xi}_i^2 + \dot{\eta}_i^2) + U, \tag{1}$$

where the potential function is*

$$U = \frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}}. \tag{2}$$

Here, we use the notation (0, 1, 2) rather than the more standard notation (1, 2, 3) because this will be more convenient later for the introduction of the heliocentric coordinates (see Figure 1). The first integrals of the center of mass can be used to reduce the number of degrees of freedom from 6 to 4 units. A convenient practical way of

* In some of the following sections, we use the notation r_1 for r_{01} and r_2 for r_{02}.

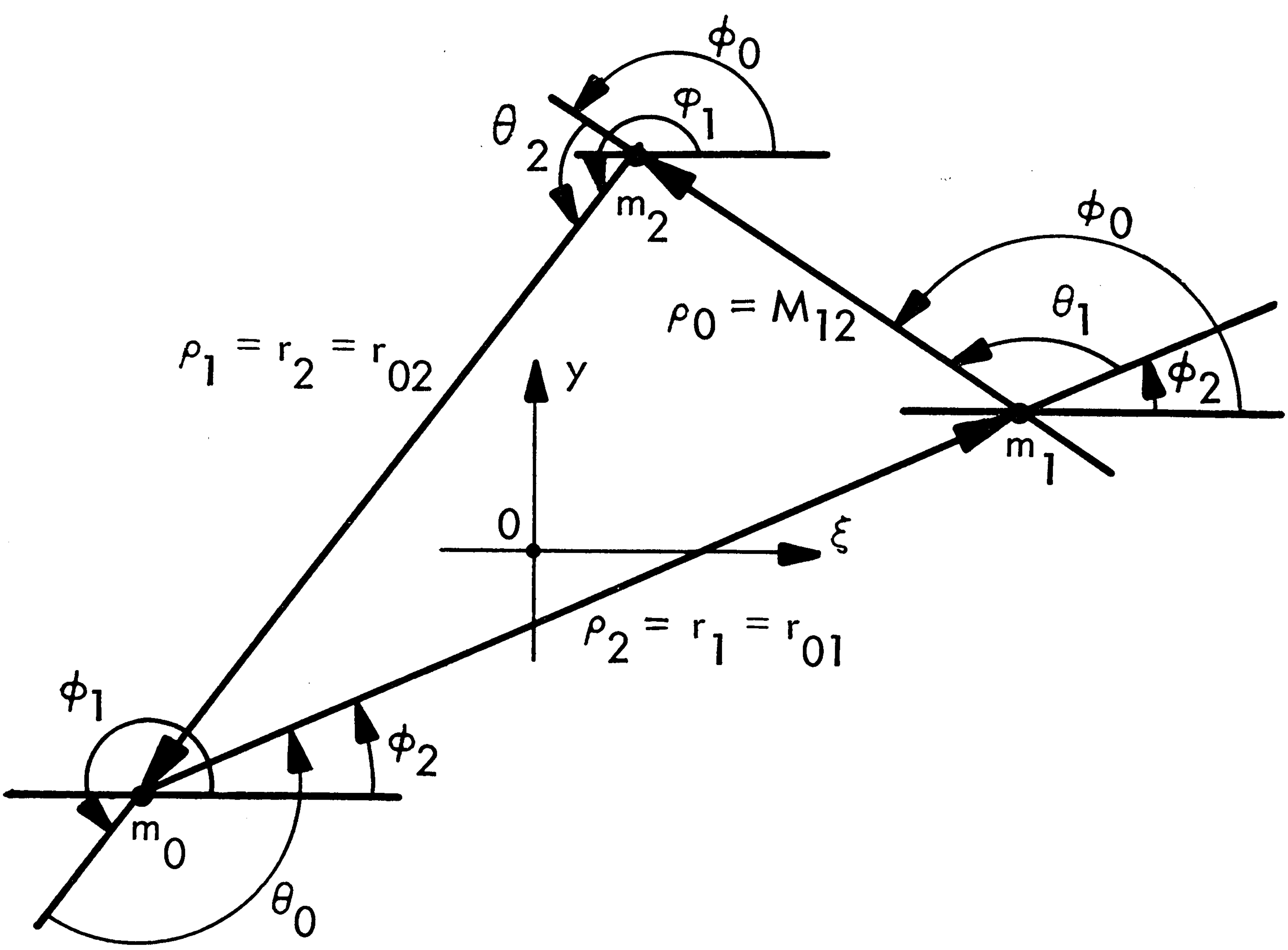

Fig. 1. The barycentric coordinate system.

reducing the number of degrees of freedom from 6 to 4 is by using the heliocentric coordinate system as will be shown later.

Let us use a simple vector notation to express the equations of motion. The barycentric position vectors are $\mathbf{r}_i$ ($i=0, 1, 2$) and the corresponding equations of motion are the twelfth-order system.

$$\ddot{\mathbf{r}}_i = -m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3} - m_k \frac{\mathbf{r}_i - \mathbf{r}_k}{r_{ik}^3}, \tag{3}$$

where i, j, k take on the values 0, 1, 2 in a circular permutation. The fact that the coordinates are barycentric can be expressed by the constraint

$$\sum_{i=0}^{2} m_i \mathbf{r}_i = 0. \tag{4}$$

The above system (3) of order 12 can be easily reduced to a system of order 8 by using the heliocentric coordinates. However, we will first show (in the next four sections) how this system can be reduced to order 6 by using the relative barycentric coordinates. Later we will also show that a similar reduction to order 6 can be made with the use of heliocentric coordinates.

3. The Equations of Relative Motion

If we are only interested in the relative positions of the three masses, it is then indicated to use as variables the three relative position vectors $\boldsymbol{\varrho}_i$ defined by

$$\boldsymbol{\varrho}_i = \mathbf{r}_k - \mathbf{r}_j \tag{5}$$

and constrained by

$$\sum_i \boldsymbol{\varrho}_i = 0. \tag{6}$$

The barycentric coordinates $\mathbf{r}_i$ can be derived from the relative coordinates $\boldsymbol{\varrho}_i$ by the linear relations

$$\mathbf{r}_i = \frac{(m_k\boldsymbol{\varrho}_j - m_j\boldsymbol{\varrho}_k)}{m}, \tag{7}$$

where m is the total mass $(m_0+m_1+m_2)$ of the system. It is now easy to obtain the relative equations of motion for the three particles: take the second derivatives of (5) and substitute (3) in the right-hand sides. We obtain the remarkably simple result

$$\ddot{\boldsymbol{\varrho}}_i = -m\frac{\boldsymbol{\varrho}_i}{\varrho_i^3} + m_i \sum_{j=0}^{2} \frac{\boldsymbol{\varrho}_j}{\varrho_j^3}. \tag{8}$$

The quantities in the denominator are the distances between particles: $\varrho_i = |\boldsymbol{\varrho}_i| = r_{kj}$.

In what follows we will develop the differential equations for the lengths of the three sides of the triangle. In other words, we will show that the second-order derivatives $\ddot{\varrho}_0$, $\ddot{\varrho}_1$ and $\ddot{\varrho}_2$ can be expressed in terms of ϱ_0, ϱ_1, ϱ_2, their first derivatives and the angular momentum only. This sixth-order system of differential equations represents the so-called relative three-body problem. Before we go into details, we collect some important geometric relations in the following section.

4. Some Preliminary Definitions

We will represent by θ_j the three exterior angles of the triangle; $(j=0, 1, 2)$. In other words, θ_j is the angle between the vectors $m_i m_j$ and $m_j m_k$ (see Figure 1). We have then the well-known laws of sines and cosines:

$$\varrho_i^2 = \varrho_j^2 + \varrho_k^2 - 2\varrho_j\varrho_k \cos\theta_i, \tag{9}$$

$$\cos\theta_i = \frac{(\varrho_i^2 - \varrho_j^2 - \varrho_k^2)}{(2\varrho_j\varrho_k)}, \tag{10}$$

$$\sin\theta_i = \frac{2\Delta\varrho_i}{(\varrho_0\varrho_1\varrho_2)}, \tag{11}$$

where Δ is the area of the triangle.

We will also use the three angles ϕ_i of the vectors $\boldsymbol{\varrho}_i$ with the x-axis. The ξ and η components of the vector $\boldsymbol{\varrho}_i$ are then (see Figure 1):

$$\xi_k - \xi_j = \varrho_i \cos \phi_i, \tag{12a}$$

$$\eta_k - \eta_j = \varrho_i \sin \phi_i. \tag{12b}$$

The constraints (6) between the three vectors $\boldsymbol{\varrho}_i$ can now be written as

$$\sum_i \varrho_i \cos \phi_i = 0, \tag{13a}$$

$$\sum_i \varrho_i \sin \phi_i = 0. \tag{13b}$$

We also have the following relations between ϕ_i and θ_i, (except eventually for a difference of 2π):

$$\theta_i = \phi_k - \phi_j. \tag{14}$$

The angular momentum integral is well known to be the constant

$$\sum_i m_i(\xi_i\dot{\eta}_i - \eta_i\dot{\xi}_i) = C, \tag{15}$$

or in terms of the variables ϱ_i and ϕ_i

$$\sum_i n_i\varrho_i^2\dot{\phi}_i = C, \tag{16}$$

where the n_i are the associated masses m_jm_k/m (see Broucke and Lass, 1973). We will see that the most important result of the present section will be an expression for the derivatives $\dot{\phi}_i$ in terms of the distances ϱ_i and their derivatives. In order to obtain this expression, we first differentiate (12a) and (12b) and we join the angular momentum integral to the result

$$\sum_i \varrho_i \cos \phi_i \cdot \dot{\phi}_i = -\sum_i \dot{\varrho}_i \sin \phi_i \tag{17a}$$

$$\sum_i \varrho_i \sin \phi_i \cdot \dot{\phi}_i = +\sum_i \dot{\varrho}_i \cos \phi_i \tag{17b}$$

$$\sum_i n_i\varrho_i^2\dot{\phi}_i = C. \tag{17c}$$

We consider the above Equations (17) as a linear system with three equations and three unknowns $\dot{\phi}_0, \dot{\phi}_1, \dot{\phi}_2$. By solving this system we obtain the following expressions for the $\dot{\phi}_i$

$$\begin{aligned}\varrho_i\dot{\phi}_i\delta = {} & C \sin \theta_i + n_j\varrho_j[\dot{\varrho}_i \cos \theta_j + \dot{\varrho}_j \cos \theta_i + \dot{\varrho}_k] - \\ & - n_k\varrho_k[\dot{\varrho}_i \cos \theta_k + \dot{\varrho}_k \cos \theta_i + \dot{\varrho}_j],\end{aligned} \tag{18}$$

where as usual i, j, k are a circular permutation of 0, 1, 2 and where the factor δ is the determinant of the system (17)

$$\delta = \frac{2J\Delta}{(\varrho_0\varrho_1\varrho_2)}. \tag{19}$$

Here, the symbol J is used for the polar moment of inertia

$$J = \sum_i m_i r_i^2. \tag{20}$$

As we said before, our important result is the expression (18) for $\dot{\phi}_i$ in terms of the sides of the triangle and their derivatives. In the formula (18), the sines and cosines of the exterior angles are assumed expressed in terms of the sides ϱ_i by using (10) and (11).

5. Equations of Motion in ϱ_i

Let us start from the definition

$$\varrho_i^2 = \boldsymbol{\varrho}_i \cdot \boldsymbol{\varrho}_i, \tag{21}$$

and differentiate twice

$$\varrho_i \ddot{\varrho}_i + \dot{\varrho}_i^2 = \dot{\boldsymbol{\varrho}}_i \cdot \dot{\boldsymbol{\varrho}}_i + \boldsymbol{\varrho}_i \cdot \ddot{\boldsymbol{\varrho}}_i. \tag{22}$$

The first term on the right side of (22) is the square of the velocity

$$\dot{\boldsymbol{\varrho}}_i \cdot \dot{\boldsymbol{\varrho}}_i = v_i^2 = \dot{\varrho}_i^2 + \varrho_i^2 \dot{\phi}_i^2. \tag{23}$$

The Equation (22) can thus be written as

$$\ddot{\varrho}_i = \varrho_i \dot{\phi}_i^2 + \frac{\boldsymbol{\varrho}_i}{\varrho_i} \cdot \ddot{\boldsymbol{\varrho}}_i, \tag{24}$$

where $\dot{\phi}_i$ is given by the Equations (18). This result will be further modified by replacing $\ddot{\boldsymbol{\varrho}}_i$ by the expressions that were given in (8) and by using the classical formulas for the dot products of two vectors

$$\boldsymbol{\varrho}_i \cdot \boldsymbol{\varrho}_j = \varrho_i \varrho_j \cos \theta_k = \tfrac{1}{2}(\varrho_k^2 - \varrho_i^2 - \varrho_j^2). \tag{25}$$

We find then the following result:

$$\ddot{\varrho}_i = \frac{(\varrho_i \dot{\phi}_i)^2}{\varrho_i} - \frac{(m_j + m_k)}{\varrho_i^2} + \frac{m_i}{2\varrho_i} \left[\frac{\varrho_k^2 - \varrho_i^2 - \varrho_j^2}{\varrho_j^3} + \frac{\varrho_j^2 - \varrho_i^2 - \varrho_k^2}{\varrho_k^3} \right]. \tag{26}$$

These equations are the final form of the equations of motion of the relative three-body problem. They are a sixth-order system in the unknowns ϱ_i and $\dot{\varrho}_i$. Solving this system determines the relative positions of the three particles of the three-body problem. The quantity $\dot{\phi}_i$ in (26) is expressed in terms of ϱ_i, $\dot{\varrho}_i$ by means of the Equations (10) and (18).

6. The Absolute Positions of the Particles

It is interesting to show that, whenever the *relative* positions of m_0, m_1 and m_2 have been obtained by integrating the differential equations of the previous section, it is then possible to obtain the *absolute* positions of m_0, m_1 and m_2 by a single additional quadrature or by integrating a single additional first-order differential equation. The only new variable that has to be added to the system is an angle ϕ, defined by (see Deprit and Delie, 1961, page 8)

$$\phi = \tfrac{1}{3}(\phi_0 + \phi_1 + \phi_2). \tag{27}$$

This definition implies the following simple relation between the angles ϕ, ϕ_i and θ_i:

$$\phi_i = \phi + \tfrac{1}{3}(\theta_j - \theta_k). \tag{28}$$

It is now a matter of relatively simple algebra to obtain the derivative of ϕ: take the derivative in the Equation (27)

$$\dot{\phi} = \tfrac{1}{3}(\dot{\phi}_0 + \dot{\phi}_1 + \dot{\phi}_2), \tag{29}$$

and replace the angles $\dot{\phi}_i$ by the expressions which have been obtained in (18). After some lengthy manipulations we find

$$\dot{\phi} = \frac{C}{J} - \sum_i A_i \dot{\varrho}_i, \tag{30}$$

where the three coefficients A_i are functions of the distances ϱ_i between the particles only (Deprit and Delie, 1961, page 9)

$$A_i = \frac{1}{12J\Delta\varrho_i}\left[(3n_i\varrho_i^2 - J)(\varrho_j^2 - \varrho_k^2) + 3\varrho_i^2(n_j\varrho_j^2 - n_k\varrho_k^2)\right]. \tag{31}$$

We see thus that not only the relative equations of motion (26) are completely free of angular quantities in their right-side member but also the differential Equation (30) for ϕ. Only the distances and their first derivatives are present. The present developments also show that the planar three-body problem has been reduced here to a system of order seven; more precisely to a system of order six (the relative three-body problem) and one quadrature. This result is an important property of the three-body problem in relation with the existence of relative and absolute periodic solutions. We defer the detailed discussion of this problem to Section 11, as we first want to develop the heliocentric reduction to order 6 of the three-body problem (Sections 7, 8, 9 and 10).

7. Heliocentric Coordinates

We will now introduce a new system of coordinates centered at m_0 and refer the two other particles m_1 and m_2 to m_0. The coordinates of m_i ($i=1, 2$) relative to m_0 will be designated by (x_i, y_i). Note that the new coordinate system has a fixed orientation, but it is not inertial, due to its moving origin. The new coordinates (x_i, y_i) will be called heliocentric for convenience, as this is often done in the study of the solar system where m_0 represents the Sun and m_i the planets.

The transformation equations are given here only for the x-components (the y-equations being completely similar).

$$x_i = \xi_i - \xi_0; \qquad i = 1, 2; \tag{32a}$$

$$0 = \sum_{i=0}^{2} m_i \xi_i. \tag{32b}$$

The last of the two equations is one of the integrals of the center of mass. The above system forms a linear system in the three coordinates ξ_i and can obviously be solved to give the barycentric coordinates as functions of the heliocentric ones (see Wintner, pages 257–258).

$$\xi_0 = -\frac{1}{m} \sum_{i=1}^{2} m_i x_i, \tag{33a}$$

$$\xi_j = x_j - \frac{1}{m} \sum_{i=1}^{2} m_i x_i, \qquad j = 1, 2. \tag{33b}$$

The Lagrangian in heliocentric coordinates has the following form, obtained by substituting (33) in (1):

$$\mathscr{L} = \frac{1}{2m} [m_1(m_0 + m_2)(\dot{x}_1^2 + \dot{y}_1^2) + m_2(m_0 + m_1)(\dot{x}_2^2 + \dot{y}_2^2)] - \\ - \frac{m_1 m_2}{m} (\dot{x}_1 \dot{x}_2 + \dot{y}_1 \dot{y}_2) + U. \tag{34}$$

The heliocentric equations of motion derived from the Lagrangian (34) are well known to be

$$\ddot{x}_i = -(m_0 + m_i) \frac{x_i}{r_i^3} + m_j \left(\frac{x_j - x_i}{r_{ij}^3} - \frac{x_j}{r_j^3} \right), \tag{35}$$

where $i = 1, 2$ and $j \neq i$. The equation in y is similar. The numerical results that will be described later have been obtained by numerically integrating the Equations (35) with the well-known recurrent power series techniques (Broucke, 1971).

8. Heliocentric Polar Coordinates

We will now introduce polar coordinates (r_i, ϕ_i), $(i = 1, 2)$ for each one of the masses m_i, with the usual definitions (see Figure 2).*

$$x_i = r_i \cos \phi_i, \\ y_i = r_i \sin \phi_i, \qquad i = 1, 2. \tag{36}$$

* The angles ϕ_1 and ϕ_2 are not the same as the angles ϕ_1 and ϕ_2 used in Figure 1 and Sections 4 through 7. No confusion will result, as the formulations of Sections 4 through 7 will not be used in what follows.

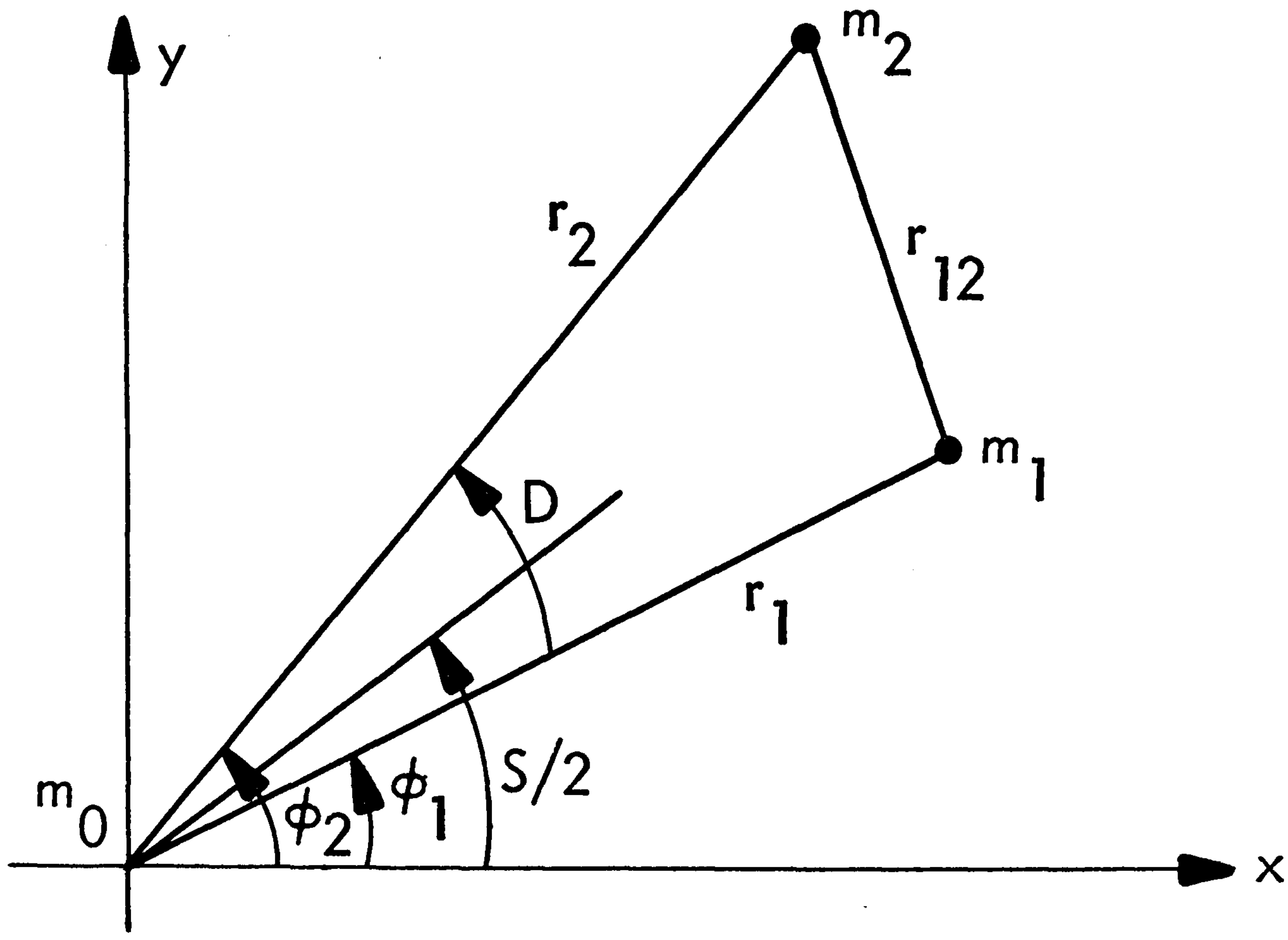

Fig. 2. The heliocentric coordinate system.

We will see later that there is a serious advantage in using these polar coordinates. In fact, this will help us to detect an ignorable (cyclic) coordinate in the new Lagrangian. The transformation of the Lagrangian (36) is a matter of elementary algebra.

$$\begin{aligned}\mathscr{L} = & \frac{1}{2m}\left[m_1(m_0 + m_2)(\dot r_1^2 + r_1^2\dot\phi_1^2) + m_2(m_0 + m_1)(\dot r_2^2 + r_2^2\dot\phi_2^2)\right] + U - \\ & - \frac{m_1 m_2}{m}\left[(\dot r_1 \dot r_2 + r_1 r_2 \dot\phi_1\dot\phi_2)\cos(\phi_2 - \phi_1) + \right. \\ & \left. + (r_1\dot r_2\dot\phi_1 - r_2\dot r_1\dot\phi_2)\sin(\phi_2 - \phi_1)\right]\end{aligned} \tag{37}$$

Here, the potential function U is still given by the expression (2); however, we assume that the distance r_{12} is expressed in terms of the polar coordinates (r_i, ϕ_i).

$$r_{12}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1). \tag{38}$$

9. Heliocentric Sum-Difference Coordinates

We see now that the new Lagrangian (37) contains the angles ϕ_2 and ϕ_1 only in the form of a difference $\phi_2 - \phi_1$. We take advantage of this observation to replace ϕ_1 and ϕ_2 by two new coordinates D and S:

$$D = \phi_2 - \phi_1, \tag{39a}$$

$$S = \phi_2 + \phi_1. \tag{39b}$$

We obtain then a new equivalent Lagrangian with 4 degrees of freedom and with variables r_1, r_2, D and S. However, only the difference D is present in the Lagrangian, but not the sum S. In other words, S is an ignorable or cyclic coordinate and it generates the first integral

$$\frac{\partial\mathscr{L}}{\partial\dot S} = \frac{1}{2}\left[\frac{\partial\mathscr{L}}{\partial\dot\phi_1} + \frac{\partial\mathscr{L}}{\partial\dot\phi_2}\right] = \text{const.} \tag{40}$$

Some elementary algebraic operations also show that the integral (40) is in fact the angular momentum integral (written in polar coordinates):

$$\begin{aligned} & m_1(m_0 + m_2)r_1^2\dot{\phi}_1 + m_2(m_0 + m_1)r_2^2\dot{\phi}_2 - \\ & \quad - m_1m_2[(r_1\dot{r}_2 - \dot{r}_1r_2)\sin(\phi_2 - \phi_1) + r_1r_2\dot{S}\cos(\phi_2 - \phi_1)] = mC. \end{aligned} \tag{41}$$

10. The Relative and the Absolute Three-Body Problem

It is easy to see the simple geometrical meaning of the two angles D and S. First of all, D is the inner angle of the triangle m_0, m_1, m_2 at m_0. The angle D and the two sides r_1 and r_2 completely determine all of the elements of the triangle. In other words, they determine the relative positions of the three particles.

The angle S on the other hand determines the position of the triangle with respect to the coordinate system. This is easily verified because $S/2$ is the angle between the x-axis and the bisectrix of r_1 and r_2. As was said before, the angle S is an ignorable variable. This fact can thus be used to reduce the number of degrees of freedom of the Lagrangian (37) from 4 to 3. First of all, $\dot{\phi}_1$ and $\dot{\phi}_2$ have to be replaced by $\dot{D}$ and $\dot{S}$ by using

$$\dot{\phi}_2 = (\dot{S} + \dot{D})/2, \tag{42a}$$

$$\dot{\phi}_1 = (\dot{S} - \dot{D})/2. \tag{42b}$$

Next, the derivative $\dot{S}$ can be eliminated from the Lagrangian by solving the first integral (41) for $\dot{S}$ and then constructing the so-called Routhian in only three coordinates (r_1, r_2, D) see (Whittaker, page 55).

The system of equations of motion (of order 6) derived from this Routhian defines the *relative planar three-body problem*. It is thus represented by three second-order differential equations in the three coordinates (r_1, r_2, D) defining only the relative positions of the three particles. This is a conservative Hamiltonian (or Lagrangian) system having only one first integral (the energy integral). The constant C of the angular momentum is to be considered as a parameter which is present in the equations of motion.

The coordinates r_1, r_2 and D do not contain sufficient information to determine the absolute positions relative to a coordinate system with fixed orientation. The supplementary angle S has to be computed for this purpose. The angle S can be obtained by a simple quadrature after the integration of the system in r_1, r_2 and D has been completed. To perform this quadrature, (41) has to be solved for $\dot{S}$. Alternately, we can consider the three second-order differential equations in r_1, r_2 and D, together with the first-order equation in S, as a single simultaneous system, of order seven. This *seventh-order system* of differential equations defines the *absolute planar three-body problem.*

In the light of what has been said above, we may represent the equations of motion of the planar three-body problem in the following symbolic form:

$$\boxed{\begin{array}{l} \ddot{r}_1 = f_1(r_i, D, \dot{r}_i, \dot{D}, m_j, C) \\ \ddot{r}_2 = f_2(r_i, D, \dot{r}_i, \dot{D}, m_j, C) \\ \ddot{D} = f_3(r_i, D, \dot{r}_i, \dot{D}, m_j, C) \\ \hline \dot{S} = f_4(r_i, D, \dot{r}_i, \dot{D}, m_j, C) \end{array}} \qquad \begin{array}{r} (43a) \\ \\ (43b) \end{array}$$

where $i=1, 2$ and m_j stands for m_0, m_1 and m_2. The equations in the upper rectangle define the *relative* three-body problem while the equations in the complete rectangle represent the *absolute* three-body problem. The Equation (43b) for $\dot{S}$ is the angular momentum integral (41).

11. Absolute and Relative Periodic Solutions

It is a completely different task to find periodic solutions for the relative three-body problem and for the absolute three-body problem. Relative periodic solutions will consist of six periodic functions of time (r_1, r_2, D and their first derivatives) while absolute periodic solutions consist of seven periodic functions with the same period, as the coordinate S now has to be added. It is easy to see that the absolute periodic solutions are a subset of the set of relative periodic solutions. Let us assume that a relative periodic solution has been found with period T. In other words, this is a periodic solution of the sixth-order system (43a). By substituting these periodic functions in (43b) we obtain a *periodic* function for $\dot{S}$. Let us thus represent this periodic function by a Fourier series with constant coefficients A_0, A_i and B_i:

$$\dot{S} = A_0 + \sum_{i=1}^{\infty} (A_i \cos i\alpha t + B_i \sin i\alpha t), \tag{44}$$

where α is the mean motion $2\pi/T$. Upon term by term integration of (44), we find the following expression for the angle S:

$$S = S_0 + A_0 t + \sum_{i=1}^{\infty} \frac{1}{i\alpha} (A_i \sin i\alpha t - B_i \cos i\alpha t). \tag{45}$$

This solution generally does not result in absolute periodic motion, due to the presence of the secular term $A_0 t$. Because of this secular term, during a complete *relative* period T, the angle S has increased by a quantity $\phi = A_0 T$; (this angle is not the same as the one defined in (27)).

There will be *absolute* periodic motion only in the case where the angle $\phi = A_0 T$ is commensurable with 2π:

$$\phi = A_0 T = \frac{p}{q} 2\pi. \tag{46}$$

By integrating such a periodic solution for q complete relative periods, we obtain an absolute periodic solution with absolute period qT. We will later show several numerical examples of this principle, corresponding to small values of p and q.

In general, during a complete period of a relative periodic solution, the whole system rotates by the angle $\phi = A_0 T$, called the rotation angle of the solution. It would thus be appropriate to represent these periodic solutions in a uniformly rotating coordinate system, with angular velocity $A_0 = \phi/T$. The angular velocity A_0 defines the natural rotating coordinate system for the periodic solution under consideration. Let us note that this angular velocity is different for each relative periodic solution. We also note that the rotation angle $\phi = A_0 T$ can be considered as an intrinsic parameter of the periodic solution. In particular, it is independent of the scaling of the solution.

12. Symmetric Periodic Solutions

In the present study we will discuss some numerically discovered *symmetric* periodic solutions of the planar three-body problem. Because of the large volume of numerical calculations that is involved in this research, we have restricted ourselves to *symmetric* solutions only. The symmetry properties of these solutions have been discussed in more detail by Broucke and Boggs (1975). The symmetry theorem states (in heliocentric coordinates) that

> If both particles m_1 and m_2 cross a fixed axis, passing through m_0 (at the origin), at the same time and at right angles, the orbits of m_1 and m_2 are symmetric with respect to this axis.

Whenever the conditions of this theorem are satisfied, we say that we have a symmetric intersection with the symmetry axis.

It is possible to use the symmetry theorem to find a sufficient periodicity condition (see Broucke and Boggs, 1975). The periodicity criterion has been stated for the case where the x-axis is the symmetry axis. It can easily be generalized to the case of two arbitrarily oriented symmetry axes.

> If a solution has two consecutive symmetric intersections at times $t=0$ and $t=T/2$, the solution is relative periodic with period T and the rotation angle is twice the angle between the two symmetry axes.

The symmetry theorem is expressed here in terms of heliocentric coordinates and an illustration taken from our family R of periodic solutions (see Section 13) is given in Figure 3a. The symmetry theorem is equally valid in barycentric coordinates, but all three orbits (rather than two) will have perpendicular intersections with the symmetry axis as is shown in the example taken from our family A given in Figure 3b.

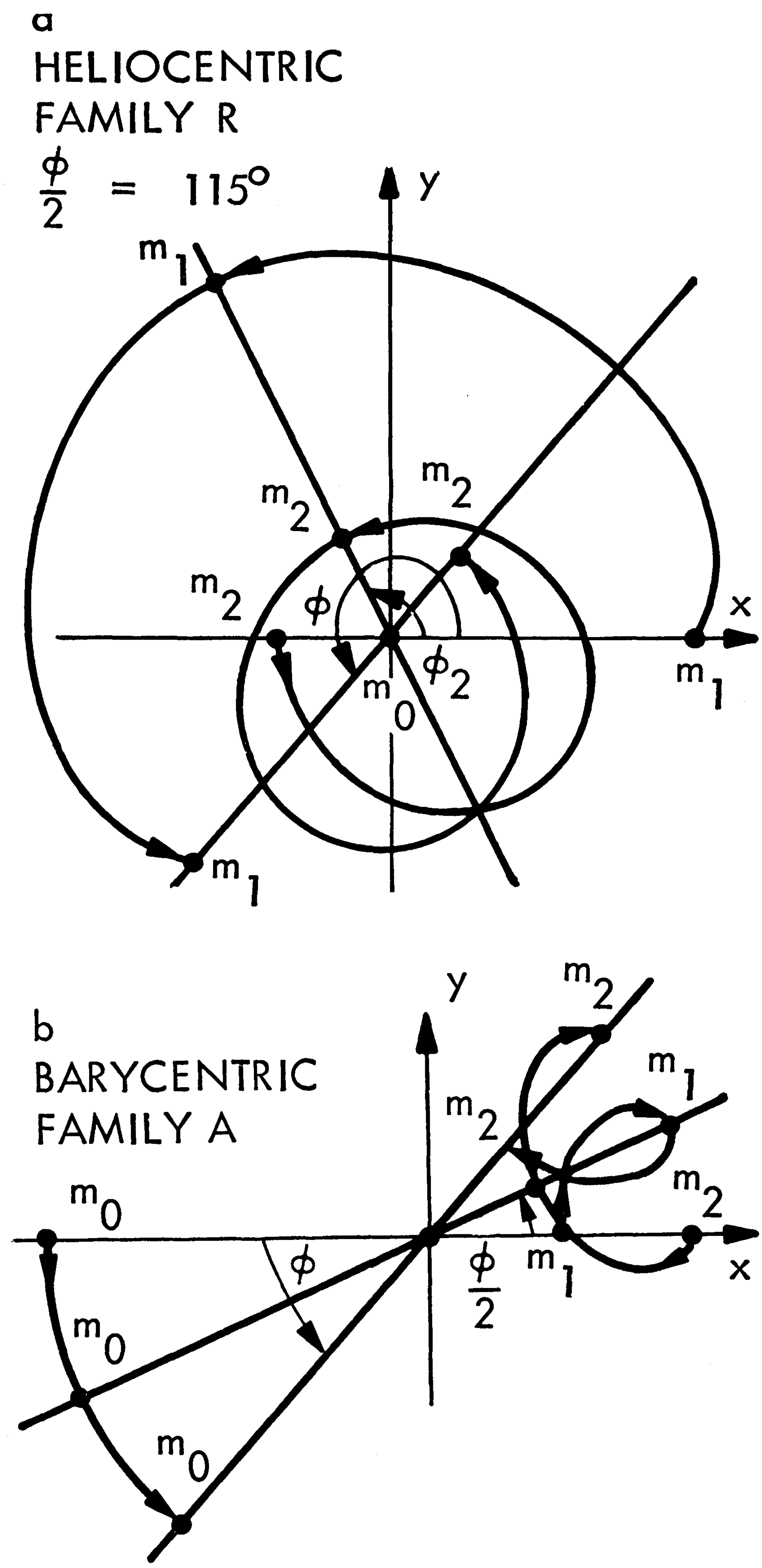

Fig. 3. The symmetry theorem in heliocentric and in barycentric coordinates.

Let us now briefly describe the numerical implementation of this periodicity condition. The orbits will only be integrated for a half period, $T/2$, from one symmetry axis to the next. The initial conditions, at the first symmetric intersection, are of the general form:

$$(x_1, y_1 = 0, \dot{x}_1 = 0, \dot{y}_1), \qquad (x_2, y_2 = 0, \dot{x}_2 = 0, \dot{y}_2).$$

Thus, four of the eight initial parameters are always zero, in order to have the initial points on the x-axis, and the velocity vectors perpendicular to it. Among the four other parameters, one may be kept fixed (x_1, for instance) in order to fix the scale of the solution (see Broucke and Boggs, 1975). There remain thus three parameters ($\dot{y}_1$, x_2, $\dot{y}_2$) to be varied in order to satisfy the periodicity criterion. It is easy to see that the periodicity criterion can be expressed analytically by exactly three conditions. The first condition states that at the second symmetric intersection the three particles must be at conjunction: m_1 and m_2 are on the same radius vector passing through the origin m_0:

$$y_1 x_2 - y_2 x_1 = 0. \tag{47}$$

The two other conditions express the right-angle requirement. The two velocity vectors must be perpendicular to the symmetry axis:

$$x_1 \dot{x}_1 + y_1 \dot{y}_1 = 0, \tag{48}$$

$$x_2 \dot{x}_2 + y_2 \dot{y}_2 = 0. \tag{49}$$

As a conclusion we see that we have three free *initial* parameters in order to satisfy three *final* conditions (47), (48) and (49). We assume here that the half-period $T/2$ is known in advance but this is generally not the case. In the numerical calculations that were performed, we left the period arbitrary. We made the numerical integrations up to a point of conjunction where the condition (47) is automatically satisfied. We reduce in this way the number of conditions from 3 to 2 (Equations (48) and (49)), and we perform differential corrections on two initial parameters, for instance, x_2 and $\dot{y}_2$. As was said above, x_1 is kept fixed in order to avoid the scale ambiguity. The parameter $\dot{y}_1$ can also be kept fixed. However, if $\dot{y}_1$ is varied and the differential corrections repeated, a new relative periodic solution is obtained. This shows that if symmetric relative periodic solutions exist, they form one-parameter families in the same way as in the circular restricted three-body problem. The differential correction process needed to find the periodic solutions requires the inversion of a 2 by 2 matrix of partial derivatives. The most economical way of obtaining these partial derivatives is by integrating two varied orbits after each nominal orbit. It is not necessary to integrate the variational equations for this purpose. Numerical integration of the variational equations is useful to obtain the monodromy matrix and the stability information.

13. Two Families of Pseudo-Circular Orbits

We describe here two families of relative periodic solutions with equal masses $m_0 = m_1 = m_2 = \frac{1}{3}$. To find the approximate initial conditions for the first member of the family, we have used the ideas of Arenstorf (1967, 1968) where the existence of certain types of periodic solutions is proved. These solutions essentially consist in double Keplerian motion: there is a pair of particles that move in approximate Keplerian motion around each other and at the same time, this binary system moves around the third particle in an approximate Keplerian motion with much larger radius. Two different families are a priori possible, according to whether the two Keplerian motions are in the same direction or not. Our numerical investigations cover both types of motion and this results in two one-parameter families of relative periodic orbits. In family A, m_0 is the isolated body and m_1, m_2 are the binary system. The motion of m_0 and the binary system around the general center of mass (the origin of the coordinate system) is direct but the components m_1 and m_2 of the binary system move around their center of mass in retrograde motion. In the second family which has been called R,* m_1 is the isolated body in a direct motion and the binary system (m_0, m_2) is also in direct motion around their own center of mass.

This situation is reminiscent of two known families of periodic orbits in the restricted three-body problem: Strömgren's Class g (Szebehely, 1967, pages 466–471) or Broucke's Class BD (Broucke, 1968, page 41) correspond to our present family R. On the other hand, Strömgren's Class f of retrograde orbits (Szebehely, 1967, pages 463–466) or Broucke's Class A_1 (Broucke, 1968, page 40) are the analogue of our present family A. However, at this point this is simply an analogy rather than a connection or transition between the families. We will mention later some real connections between our families and the known orbits of the restricted problem.

As both families are analytical continuations of circular Keplerian motions, it is easy to obtain approximate initial conditions to start these two families. If we assume that a binary system m_1+m_2 moves around m_0 on a circle with preassigned radius a, the velocity V_a in this circular motion is given by

$$V_a^2 = \frac{m_0 + m_1 + m_2}{a}.$$

In the same way, if we assume circular motion of m_2 around m_1, with arbitrary radius b, ($b \ll a$), the circular velocity V_b is given by

$$V_b^2 = \frac{m_1 + m_2}{b}.$$

The four Keplerian quantities a, b, V_a, V_b contain all the information that is needed to form the approximate initial conditions of the numerical integration of the exact three-body equations of motion. These initial conditions are improved by the usual linear differential correction procedures.

If we assume that b is much smaller than a, the period T of such a relative periodic solution will also be small, and so will the rotation angle ϕ. In fact, when b tends to zero, T and ϕ also tend to zero. The initial orbit that has been computed numerically corresponds to a value $\phi/2=0.058\,78$ rad for our retrograde family A and the value $\phi/2=0.1978$ rad for our direct family R. In both cases we consider the rotation angle

* The nomenclature A and R results from the fact that we have obtained several families of periodic solutions, all designated by one letter of the alphabet. Some of the other families will be documented later.

ϕ as the parameter of the family. We give $\phi/2$ in the tables of heliocentric initial conditions rather than ϕ, because only half-orbits are computed for symmetric orbits. For each family about 200 orbits were computed, but to limit space we give in Tables I and II only about 12 orbits per family, at more or less equidistant intervals in ϕ. It so happens that for both families, ϕ is monotonically increasing, but this should not be considered as a general rule. The final values of $\phi/2$ that have been reached up to now are 3.1428 for family A and 3.8868 rad for family R. The two families could be extended further, but this work is still in progress. Both families are far enough developed for the study of several important properties. In particular the range of the rotation angle ϕ is such that a large number of *absolute* periodic solutions can be extracted from both families. More precisely, 16 absolute periodic solutions have been interpolated with high precision in family A and 13 in family R. The initial conditions are in Tables III and IV.

TABLE I

Heliocentric initial conditions of family A of relative periodic orbits

x_1	$\dot{y}_1$	x_2	$\dot{y}_2$	$T/2$	$\varphi/2$
12.779 563 528	0.708 965 414	13.665 328 929	−0.158 709 982	3.141 592 654	0.065 367 975
3.337 554 139	0.935 709 961	4.293 505 538	0.095 702 925	3.141 592 654	0.424 057 748
1.947 496 097	1.060 072 707	2.992 153 817	0.243 708 801	3.141 592 654	0.824 023 787
0.723 947 921	1.392 262 949	2.175 486 553	0.619 230 096	3.141 592 654	2.039 305 674
0.299 397 148	2.023 806 412	2.264 265 773	1.164 053 051	3.141 592 654	2.928 369 964
0.281 166 859	2.087 842 671	2.275 396 922	1.206 866 276	3.141 592 654	2.964 730 859
0.246 359 384	2.231 887 968	2.299 125 851	1.296 813 903	3.141 592 654	3.028 711 256
0.219 966 727	2.365 624 237	2.319 847 324	1.373 902 856	3.141 592 654	3.071 041 555
0.194 899 309	2.518 977 519	2.342 234 591	1.456 700 009	3.141 592 654	3.105 199 113
0.160 055 405	2.792 300 992	2.378 409 085	1.594 811 084	3.141 592 654	3.141 744 329

TABLE II

Heliocentric initial conditions of family R of relative periodic orbits

x_1	$\dot{y}_1$	x_2	$\dot{y}_2$	$T/2$	$\varphi/2$
5.884 007 452	−0.047 897 763	−0.837 181 789	−0.893 590 331	3.141 592 654	0.198 880 363
3.949 711 509	0.024 944 693	−0.810 458 734	−0.910 816 552	3.141 592 654	0.346 816 619
2.628 488 721	0.104 539 871	−0.763 570 802	−0.947 015 555	3.141 592 654	0.605 205 704
1.564 741 216	0.180 379 728	−0.640 618 480	−1.084 553 287	3.141 592 654	1.226 100 549
1.126 864 822	0.078 541 963	−0.437 071 336	−1.469 380 211	3.141 592 654	2.004 733 148
1.010 415 523	−0.074 519 903	−0.311 434 477	−1.849 074 104	3.141 592 654	2.433 151 276
0.910 107 163	−0.345 157 796	−0.181 149 814	−2.544 658 713	3.141 592 654	2.955 495 807
0.866 404 631	−0.547 041 881	−0.128 621 515	−3.070 106 005	3.141 592 654	3.218 008 559
0.819 911 082	−0.906 826 802	−0.079 497 279	−3.965 217 822	3.141 592 645	3.505 194 342
0.785 539 889	−1.463 648 761	−0.046 425 097	−5.247 907 056	3.141 592 654	3.721 287 165
0.770 364 599	−2.030 500 683	−0.030 845 353	−6.477 510 143	3.141 592 654	3.823 797 052
0.762 635 804	−2.757 603 773	−0.020 410 126	−7.999 088 844	3.141 592 654	3.886 840 178

TABLE III

Heliocentric initial conditions of 16 absolute periodic orbits of family A

x_1	$\dot{y}_1$	x_2	$\dot{y}_2$	$T/2$	p/q
0.160 231 433	2.790 692 952	2.378 210 758	1.594 022 431	3.141 592 654	1/1
1.000 000 000	1.689 116 798	1.300 821 227	0.190 647 546	3.851 082 150	1/7
1.000 000 000	1.634 630 160	1.344 911 402	0.231 949 084	3.955 178 441	1/6
1.000 000 000	1.571 443 159	1.408 590 793	0.277 741 508	4.105 508 010	1/5
1.000 000 000	1.495 533 832	1.510 089 176	0.329 531 844	4.344 968 845	1/4
1.000 000 000	1.399 222 655	1.701 970 040	0.390 233 414	4.796 709 493	1/3
1.000 000 000	1.265 852 825	2.227 877 156	0.469 644 019	6.028 134 049	1/2
1.000 000 000	1.450 658 441	1.588 297 336	0.358 469 919	9.058 538 840	2/7
1.000 000 000	1.338 820 955	1.884 122 621	0.426 138 498	10.448 244 359	2/5
1.000 000 000	1.176 872 952	3.128 698 450	0.533 630 860	16.304 771 638	2/3
1.000 000 000	1.316 108 680	1.972 417 081	0.439 493 002	16.292 577 034	3/7
1.000 000 000	1.208 241 335	2.701 147 795	0.508 159 639	21.412 303 043	3/5
1.000 000 000	1.145 060 889	3.864 798 318	0.566 656 486	29.857 306 670	3/4
1.000 000 000	1.223 349 888	2.548 687 677	0.497 321 994	27.116 185 427	4/7
1.000 000 000	1.129 973 353	4.473 155 821	0.587 118 231	46.028 999 271	4/5
1.000 000 000	1.157 691 647	3.514 995 987	0.552 307 896	45.436 082 836	5/7

TABLE IV

Heliocentric initial conditions of 13 absolute periodic orbits of family R

x_1	$\dot{y}_1$	x_2	$\dot{y}_2$	$T/2$	p/q
0.777 401 091	−0.510 832 077	−0.126 551 669	−3.081 358 822	2.613 279 379	1/1
0.777 401 091	0.214 517 863	−0.329 939 662	−1.580 023 406	2.852 382 191	1/2
0.777 401 091	0.258 226 141	−0.300 691 126	−1.554 882 578	2.780 294 241	1/3
0.777 401 091	0.234 494 465	−0.261 744 816	−1.633 510 799	2.698 036 144	1/4
0.777 401 091	0.199 270 688	−0.230 872 977	−1.724 137 060	2.633 549 487	1/5
0.777 401 091	0.162 303 686	−0.206 954 546	−1.812 692 424	2.583 704 945	1/6
0.777 401 091	0.126 080 857	−0.188 063 538	−1.896 407 206	2.544 368 993	1/7
0.777 401 091	0.061 216 289	−0.290 493 543	−1.827 632 183	5.612 508 053	2/3
0.777 401 091	0.254 184 039	−0.320 515 292	−1.538 554 162	5.648 075 401	2/5
0.777 401 091	0.249 125 087	−0.280 329 148	−1.590 658 229	5.474 118 981	2/7
0.777 401 091	0.134 877 460	−0.313 838 514	−1.701 242 021	8.510 759 659	3/5
0.777 401 091	0.246 925 212	−0.325 773 812	−1.542 454 708	8.510 195 359	3/7
0.777 401 091	0.161 627 621	−0.320 965 116	−1.658 251 145	11.381 992 217	4/7

As was said above, the orbits are nearly circular for both of these families. In both families the isolated body (m_0 in family A and m_1 in family R) follows a direct, almost circular path. Now, if we represent the motion in a coordinate system which also rotates in a direct way with the same average angular velocity, we can observe that with respect to this rotating frame, m_0 (family A) or m_1 (family R) is hardly moving. In fact, these bodies describe an orbit of very small size, visible as a point on Figures

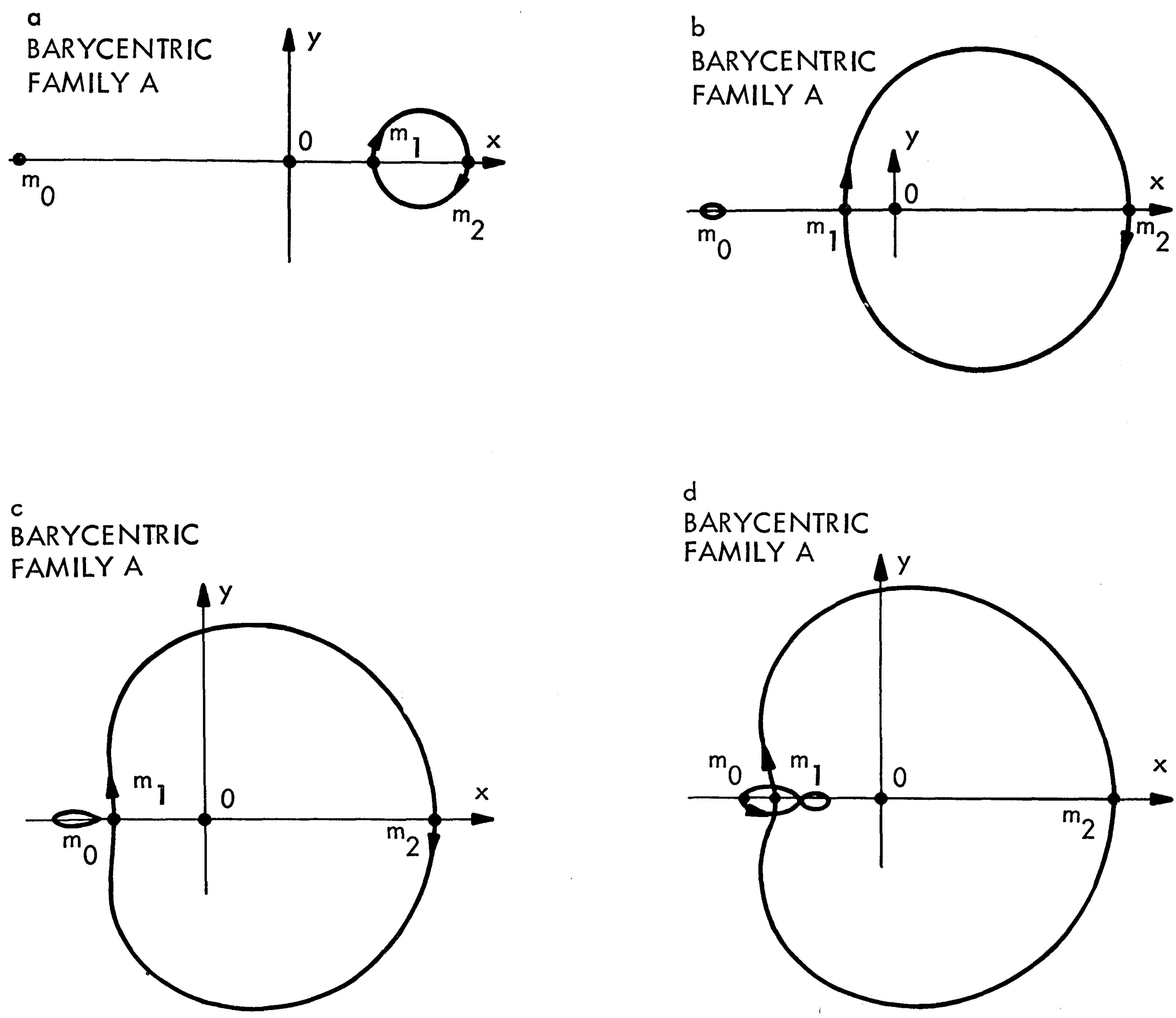

Fig. 4. Relative periodic orbits of family A in a rotating coordinate system.

4a and 5a. This fact is mostly true for the first orbits of each family. It is still true for the last orbits, although the orbits are now noticeably larger. In family A (Figure 4d), m_0 describes a small figure-eight orbit; in family R, m_1 describes a small oval path perpendicular to the x-axis (Figure 5d). Also we observe that for both families the two bodies forming the binary system travel on the same path in the rotating coordinate system. This property, which occurs frequently in the three-body problem with equal masses, has been commented on by Broucke and Boggs (1975). For both families we observe a close approach between two bodies towards the end of the family. A binary collision orbit will probably develop and a regularized program is necessary to continue these families.

Some typical orbits in the nonrotating frame of reference are given in Figures 6, 7 and 8. As stated previously, in the rotating coordinate system the two bodies forming the binary pair travel on the same curve. A similar property exists for the absolute periodic solutions with commensurability ratios p/q with *even p*. This property holds in the barycentric as well as the heliocentric nonrotating frame.

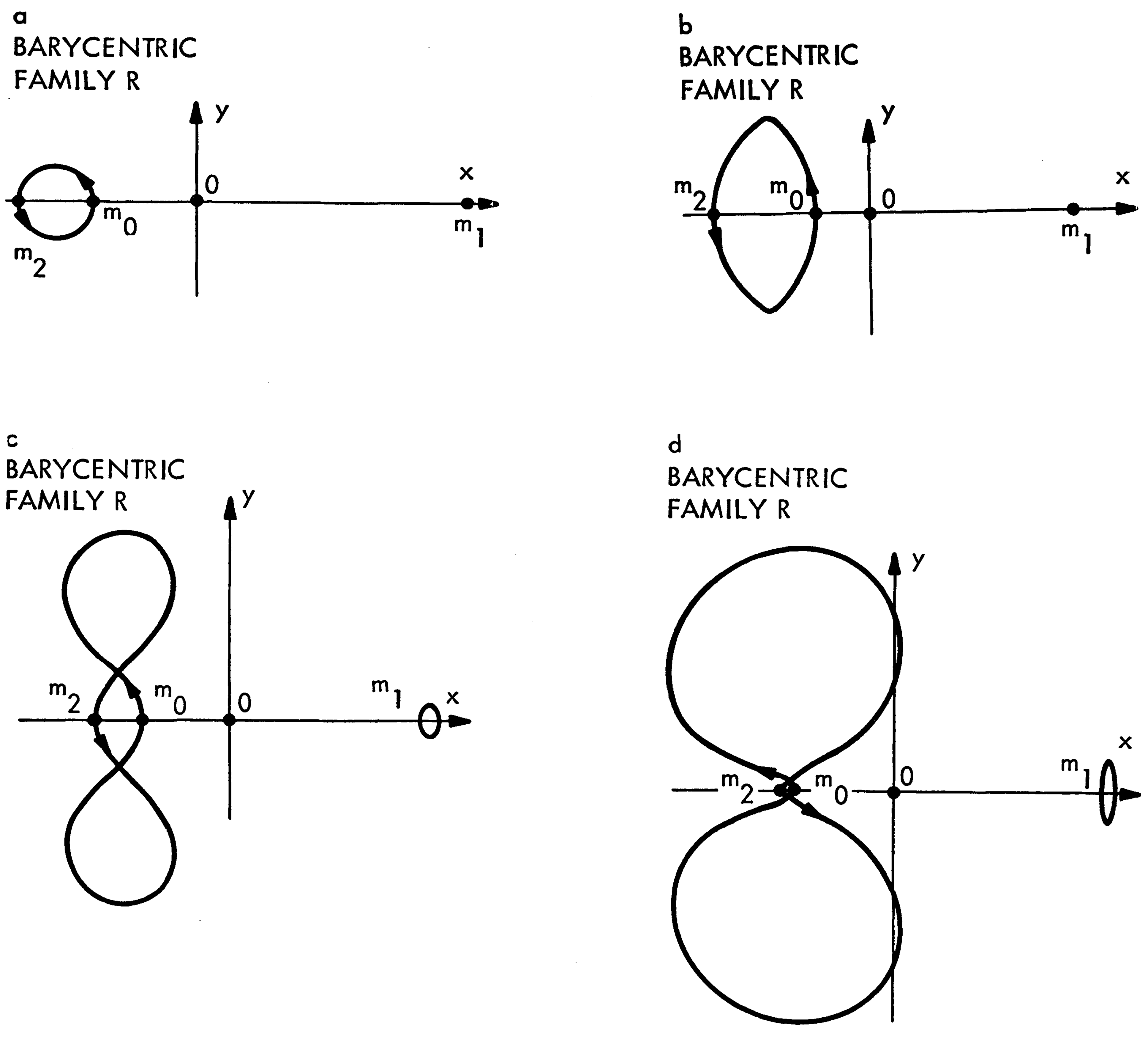

Fig. 5. Relative periodic orbits of family R in a rotating coordinate system.

Solutions 1, 6 and 40 of our previous article belong to our present family A (with commensurability ratios 1/2, 1/5 and 1/3); (Broucke and Boggs, 1975).

In both families, we have an absolute periodic solution with rotation angle 2π, or with commensurability ratio 1/1. These two solutions are remarkable illustrations of the phenomenon of gravitational interplay (Szebehely, 1971) between two or all three of the bodies. As shown in Figure 9, the 1/1 case of family R is relatively simple because this is still a case of a binary system revolving about a more or less undisturbed third body. However, the orbit with ratio 1/1 in family A is a more remarkable case because we have interaction (interplay) between the three bodies; m_1 and m_2 describe similar orbits around m_0. All three orbits are approximately ellipses of the same eccentricity. At $t=0$, m_0 and m_1 have a close approach, while at $t=T/2$, m_0 and m_2 have a close approach along a similar path (see Figures 10a and 10b). The orbits are symmetric with respect to the x-axis and the y-axis.

We also found that the orbit A. 1/1 can probably be considered as an extension of a

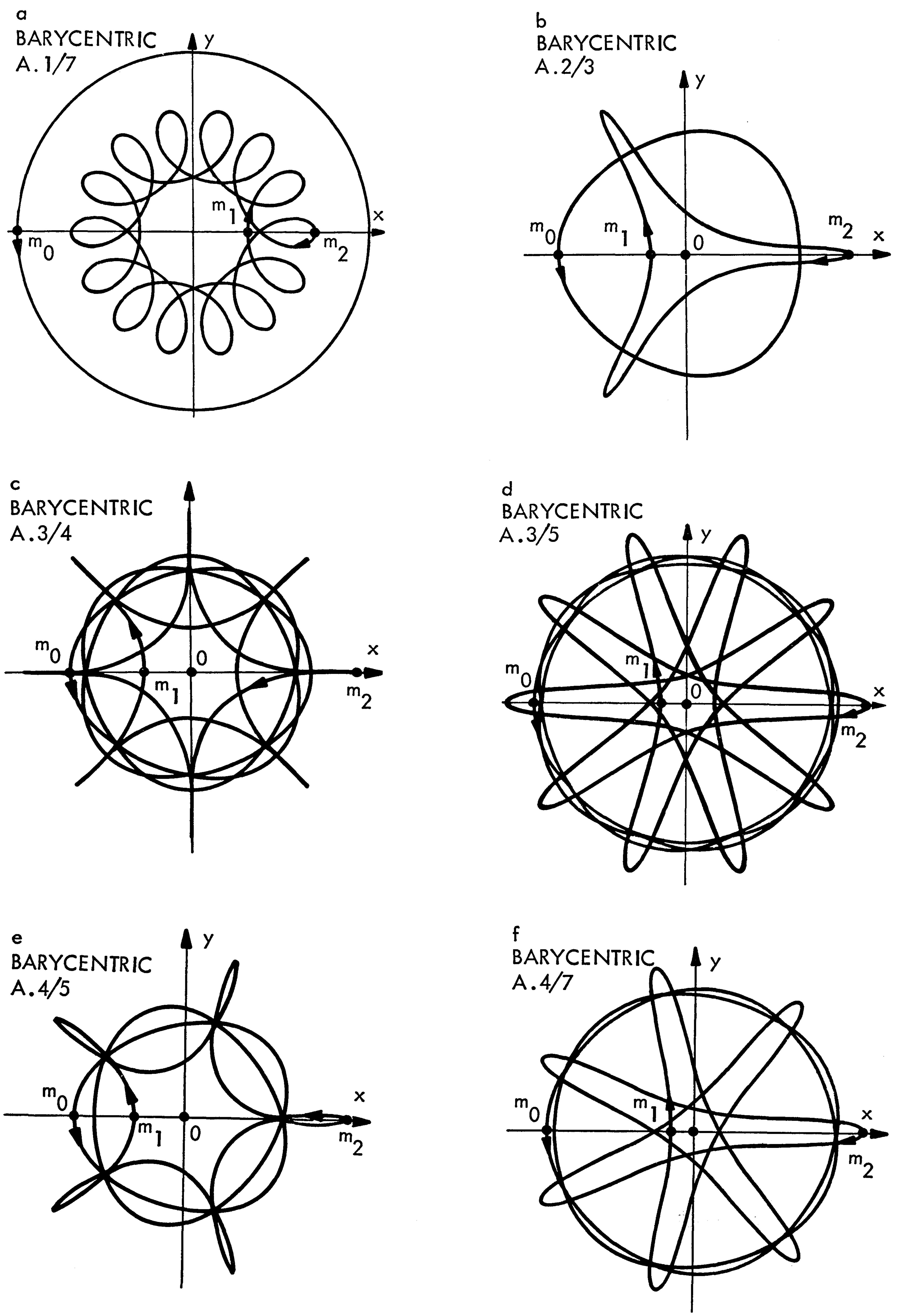

Fig. 6. Absolute periodic orbits of family A in a barycentric coordinate system.

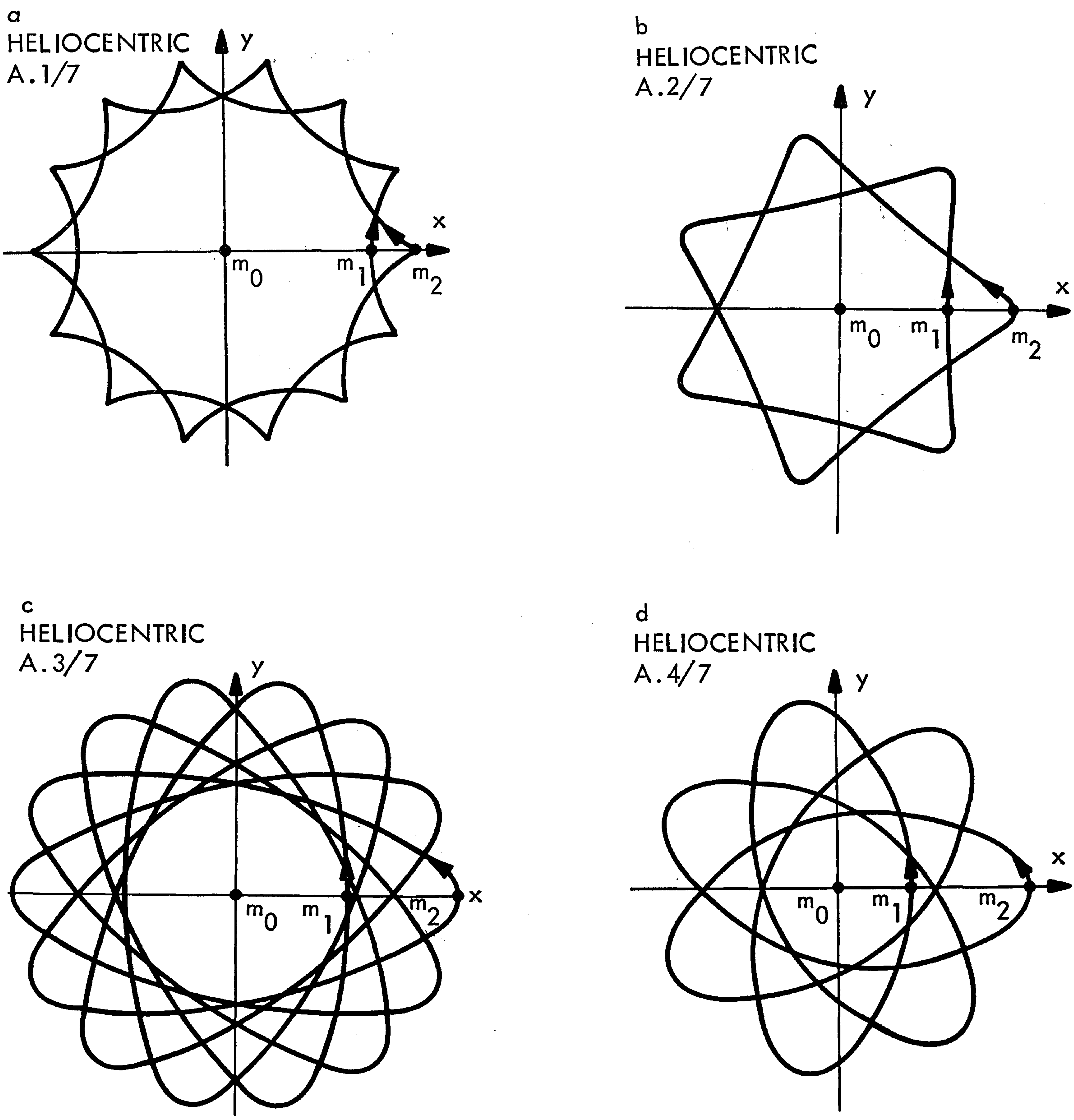

Fig. 7. Absolute periodic orbits of family A in a heliocentric coordinate system.

known orbit of the elliptic restricted three-body problem of our family $8A$ (see Broucke, 1969, page 51), which is itself an extension of class g of periodic orbits of the Strömgren problem. Indeed, by starting from the initial conditions of the elliptic problem (family $8A$, masses $m_0=\frac{1}{2}$, $m_1=\frac{1}{2}$ and $m_2=0$), we can obtain a periodic solution with masses $m_0=0.499$, $m_1=0.499$ and $m_2=0.002$ (Figures 11a and 11b), which is completely similar to the solution with three equal masses. The present solution has some other interesting properties related to the stability and the characteristic exponents. These will be described later.

It appears certain that the orbits of our family R are related to the family of solu-

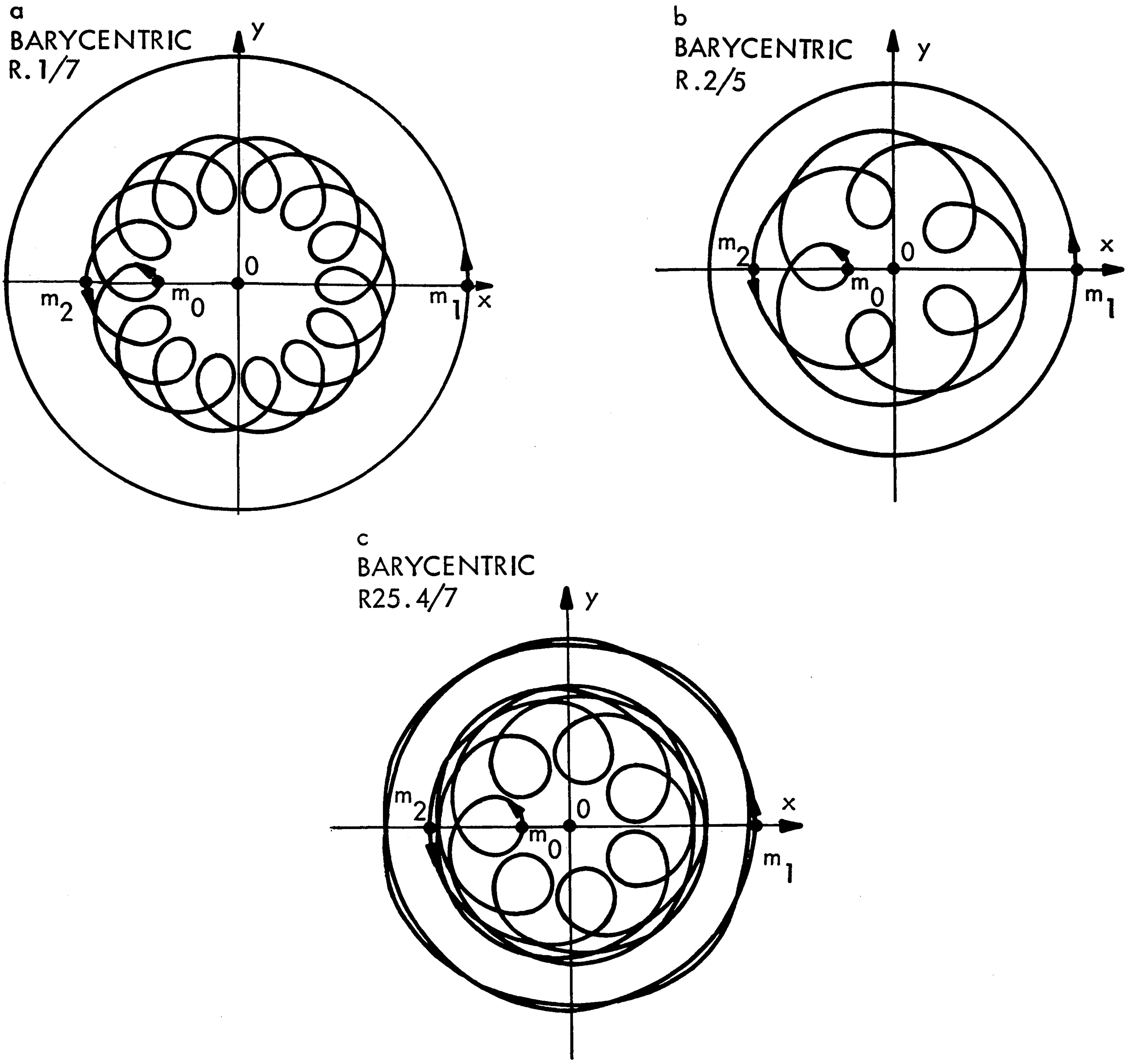

Fig. 8. Absolute periodic orbits of family R in a barycentric coordinate system.

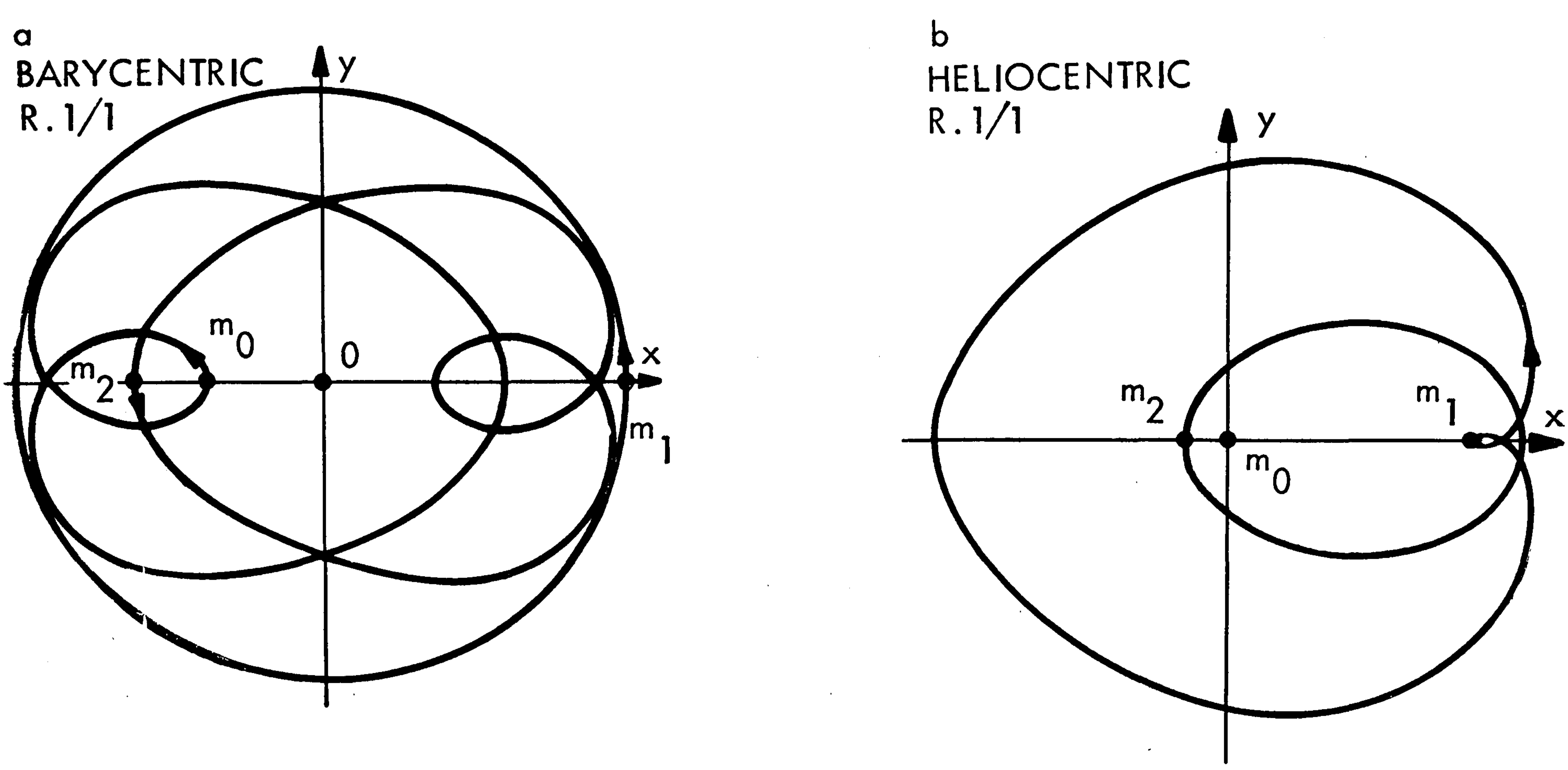

Fig. 9. The 1/1 resonance case in family R.

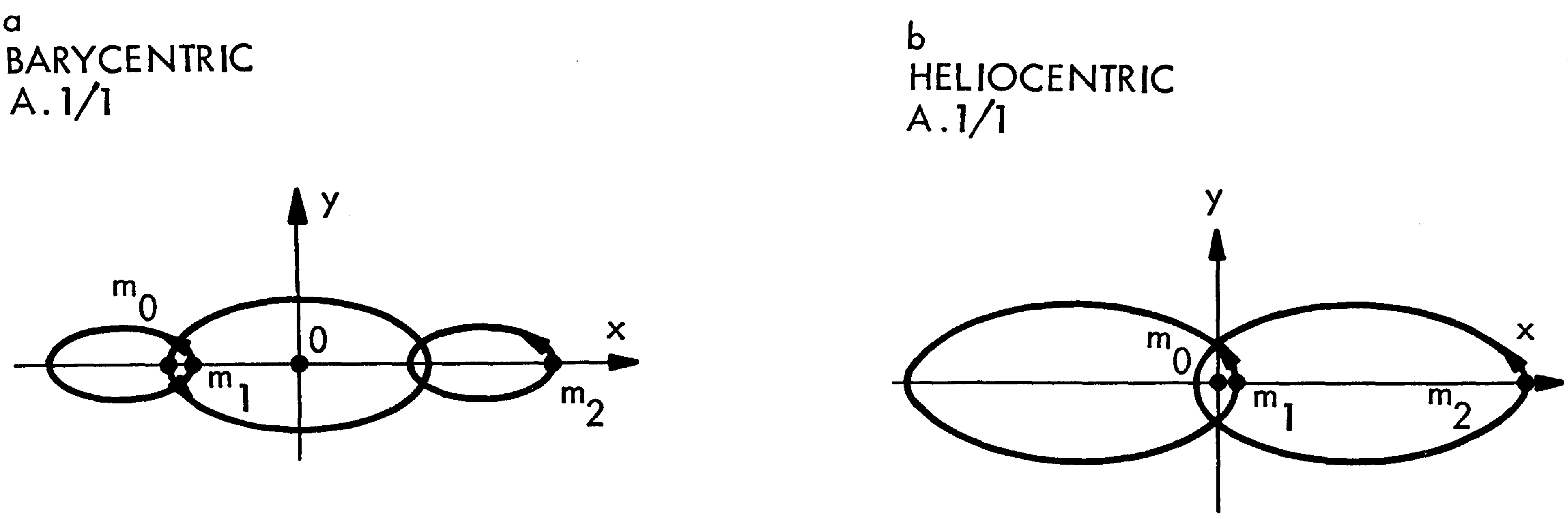

Fig. 10. The 1/1 resonance case in family A.

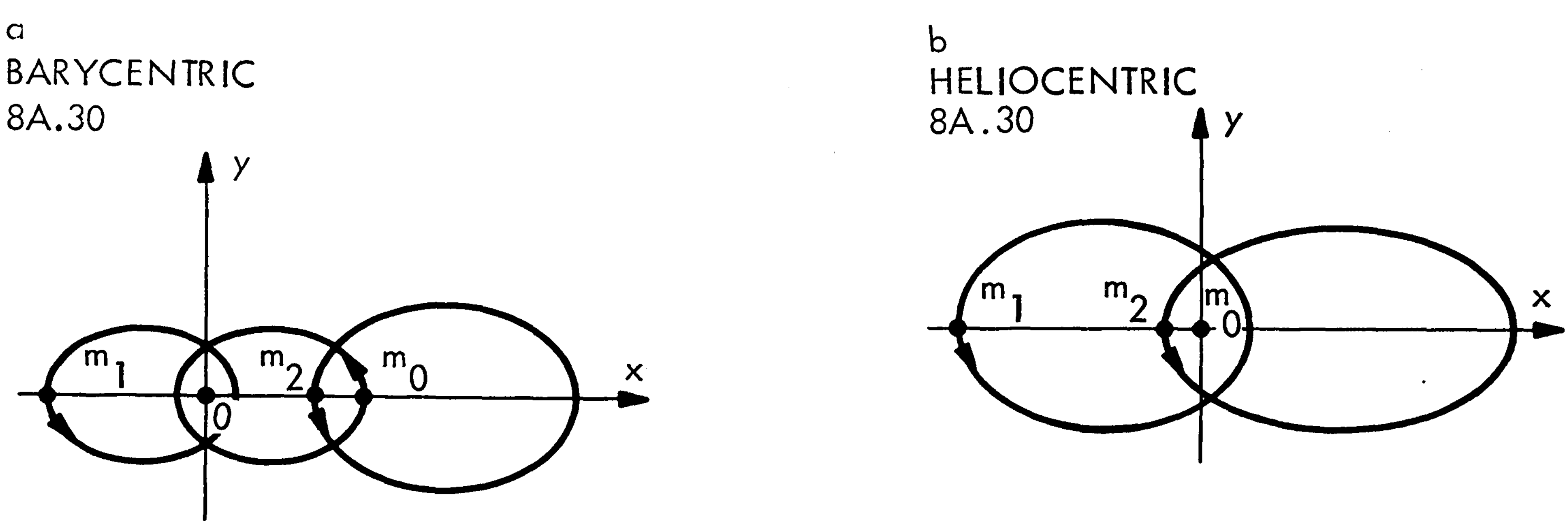

Fig. 11. The 1/1 resonance case with masses 0.499, 0.499 and 0.002.

tions recently computed by Hadjidemetriou (1975). These orbits are continuations of Strömgren's class g. In any way the transition from the restricted to the general three-body problem is a complex phenomenon and more research in this area would be appropriate.

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