Dynamical group of the relativistic Kepler problem

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A manifestly covariant relativistic generation of the Lenz vector for the relativistic Kepler problem is constructed. The corresponding dynamical group for the bound state problem is SO(4,1).

It is well known[1] that the nonrelativistic quantum mechanical Kepler problem can be treated by studying a symmetry group which is larger than the explicit SO(3) symmetry of the (relative motion) Hamiltonian

$$H=\frac{\mathbf{p}^2}{2M}+\frac{\alpha}{r}, \tag{1}$$

where α is the fine structure constant, $\mathbf{r}=\sqrt{(x_1-x_2)^2}$ is the distance between the particles, M is the reduced mass of the two body system, and $\mathbf{p}=(M_2\mathbf{p}_1-M_1\mathbf{p}_2)/(M_1+M_2)$ is the relative momentum. This larger group (for the bound states, where $H<0$) is SO(4), whose generators are the angular momentum $\mathbf{L}$ and the Lenz vector is

$$\mathbf{A}=\sqrt{-\frac{M}{2H}}\left\{\frac{1}{2M}(\mathbf{L}\times\mathbf{p}+\mathbf{p}\times\mathbf{L})+\frac{\alpha\mathbf{r}}{r}\right\}. \tag{2}$$

The vector $\mathbf{A}$ is a constant of the motion (in the corresponding classical problem it is directed along the major axis of the elliptic orbit). Its components have the commutation relations

$$[A_i,A_j]=i\epsilon_{ijk}L_k. \tag{3}$$

It was recently shown[2] that a relativistically manifestly covariant form of the Kepler problem can be constructed with (relative motion) Hamiltonian[3] [we use metric $(-+++)$]

$$K=\frac{p^\mu p_\mu}{2m}-\frac{\alpha}{\rho}, \tag{4}$$

where $p^\mu=(M_2p_1^\mu-M_1p_1^\mu)/(M_1+M_2)$, $m=M_1M_2/(M_1+M_2)$ are the relative energy momentum and reduced mass of the two body system, and $\rho=\sqrt{(\mathbf{x}_1-\mathbf{x}_2)^2-(t_1-t_2)^2}$ is the invariant (spacelike) distance between the particles.

In the nonrelativistic limit $t_1\to t_2$, so that $\rho\to r$; the relative energy $E=(M_2t_1-M_1t_2)/(M_1+M_2)$, where M_1 and M_2 are the masses of the individual particles, also vanishes and hence K goes over to the form (1).

The bound state spectrum $\{K'\}$ of the operator (4) was shown to be precisely equal to the energy spectrum of the nonrelativistic problem. Since, however, the total Hamiltonian is

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$$K_T=\frac{P^\mu P_\mu}{2M}+K', \tag{5}$$

where $P^\mu=P_1^\mu+P_2^\mu$, $M=M_1+M_2$, and K_T is a constant that can be evaluated[2] at the ionization point as $-M/2$, we see that in the center of mass frame

$$E=\sqrt{M^2+2MK'}. \tag{6}$$

If the coupling is such that the excitations K' are small compared to M, then

$$E\cong M+K'+O\left(\frac{K'^2}{M}\right), \tag{7}$$

where the last term corresponds to relativistic corrections.

This problem was solved by studying the differential equation

$$K\psi=K'\psi. \tag{8}$$

One may ask whether there exists, in the relativistic case, an analog of the Lenz vector (2). It is straightforward to show, using the commutation relations

$$[x^\mu, p^\nu]=ig^{\mu\nu}, \tag{9}$$

that the vector

$$A_\mu=\sqrt{-\frac{m}{2K}}\left\{\frac{1}{2m}\,(M_{\nu\mu}p^\nu+p^\nu M_{\nu\mu})+\frac{\alpha x_\mu}{\rho}\right\}, \tag{10}$$

where

$$M_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$$

are generators of the Lorentz group, is a constant of the motion, i.e.,

$$[A_\mu\,, K]=0. \tag{11}$$

The commutation relations of A_μ with $M_{\nu\lambda}$ are those of a vector; the relations

$$[M_{\lambda\mu},p_\nu]=-(g_{\mu\nu}p_\lambda-g_{\lambda\nu}p_\mu), \tag{12}$$

$$[M_{\lambda\mu},M_{\delta\nu}]=-i(g_{\mu\delta}M_{\lambda\nu}-g_{\lambda\nu}M_{\delta\mu}-g_{\mu\nu}M_{\lambda\delta}+g_{\delta\lambda}M_{\nu\mu}) \tag{13}$$

are helpful in calculating the commutator between components of A_μ. One finds

$$[A_\mu\,, A_\nu]=iM_{\mu\nu}. \tag{14}$$

We furthermore note that A_0 is proportional to M_{j0} and x_0, and therefore vanishes in the nonrelativistic limit. The vector A_μ is therefore a proper relativistic generalization of the Lenz vector for the two-body Kepler problem.

The relation (14), along with (13) and

$$[M_{\lambda\mu},A_\nu]=-i(g_{\mu\nu}A_\lambda-g_{\mu\nu}A_\mu) \tag{15}$$

define an algebra of the associated dynamical group. Writing these relations in terms of components, with $L_i=M_{jk}$ (ijk cyclic), we have

$$[A_i,A_j]=iL_k \quad (ijk \text{ cyclic}),$$

$$[L_i,L_j]=iL_k \quad (ijk \text{ cyclic}),$$

$$[L_i,A_j]=iA_k \quad (ijk \text{ cyclic}), \tag{16}$$

$$[A_0,A_j]=iM_{0j},$$

$$[M_{0j},A_0]=-iA_j,$$

$$[M_{0i},M_{0j}]=-iL_k \quad (ijk \text{ cyclic}). \tag{17}$$

The relations (16) and (17) define the group SO(4,1); (16) is the algebra of the SO(4) subgroup. In a representation of SO(4,1) induced on its maximal compact subgroup SO(4), the spectrum of K coincides with that of the energy spectrum of the nonrelativistic Kepler problem.[1] In such a representation, we shall require that

$$A^\mu A_\mu=\mathbf{A}^2-A^{0^2}\geqslant 0$$

so that the nonrelativistic limit is correct.

Note that the L_i have the same form as the nonrelativistic case, but

$$A_i=\sqrt{-\frac{m}{2K}}\left\{\frac{1}{2m}\,(M_{ji}p^j+p^jM_{ji})+\alpha\,\frac{x_j}{\rho}+\frac{1}{2m}\,(M_{0i}p^0+p^0M_{0i})\right\}, \tag{18}$$

i.e., the space part of the Lenz vector contains an additional term (which vanishes in the nonrelativistic limit). It is, however, an identity that

$$\epsilon_{\mu\nu\lambda\sigma}A^\nu M^{\lambda\sigma}=0, \tag{19}$$

since A_ν contains x^ν or p^ν in every term (as right factors). Taking the $\mu=0$ component, it follows that

$$\mathbf{A}\cdot\mathbf{L}=0 \tag{20}$$

in the relativistic case as well. It was shown in Ref. 4 that, for the classical Kepler problem, there exists a frame in which the relative t is zero. The problem reduces, in this case, to the form of the nonrelativistic one. Equation (20) (with the $\mathbf{A}$ and $\mathbf{L}$ of that frame) is the zero component of a four vector, which, under Lorentz transformation takes on the form (19).

There are two Casimir operators for SO(4,1). These are obtained by appending A_μ antisymmetrically as a row and column to $M_{\mu\nu}$ and constructing the independent second order and fourth order invariants. They are

$$c_1=2A^\mu A_\mu+M^{\mu\nu}M_{\mu\nu}, \tag{21}$$

$$\begin{aligned}c_2={}&M_{\mu\nu}M^\nu_\lambda M^\lambda_\gamma M^{\gamma\mu}-A_\mu A_\lambda M^\lambda_\gamma M^{\gamma\mu}-M_{\mu\nu}A^\nu A_\gamma M^{\gamma\mu}\\&-M_{\mu\nu}M^\nu_\lambda A^\lambda A^\mu-A_\nu M^\nu_\lambda M^\lambda_\gamma A^\gamma+A_\mu A_\lambda A^\lambda A^\nu+A_\nu A^\nu A_\lambda A^\lambda.\end{aligned} \tag{22}$$

The degeneracy of the levels is the same as for the nonrelativistic problem in the SO(4) subgroup, but in SO(4,1) there is additional multiplicity. The general structure of the representations and in geometrical meaning of the relativistic Lenz vector will be discussed in a succeeding publication.

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